Game Theory: Continuous Strategies and Duopoly

Adaptive and Cooperative Algorithms (ECE 457A)

ECE, MME, and MSCI Departments,
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Continuous Strategies

Continuous Strategies

- **Continuous strategies** refer to when the action space in continuous.
- It is slightly different from **mixed strategy** in the sense that mixed strategy is continuous as an interpolated probability between selection of pure actions but a continuous strategy has continuous actions.

Duopoly

Duopoly

- **Duopoly** means two companies are competing in the market [1, 2].
- **Monopoly** means one company dominates the market.
- Three most well-known models of a duopoly are:
  - Cournot competition
  - Stackelberg competition
  - Bertrand competition
- These games have **continuous strategies** where the actions are continuous.

Cournot competition

Cournot competition

- **Cournot** competition, proposed in 1838 [3]:
  - continuous strategy space
  - duopoly in which two firms choose output levels (quantities) in competition with each other
  - simultaneous moves
- every company makes q_i quantities
- actions: choosing quantities q_1 and q_2 simultaneously
- marginal cost per making quantity: $c = 12$
- demand (total generated quantity): $Q = q_1 + q_2$
- profit per quantity: $p(Q) = 120 - q_1 - q_2$
- the payoffs of players:

$$\pi_1 = (120 - q_1 - q_2)q_1 - cq_1 = (120 - c)q_1 - q_1^2 - q_1 q_2$$
$$\pi_2 = (120 - q_1 - q_2)q_2 - cq_2 = (120 - c)q_2 - q_2^2 - q_1 q_2$$

Cournot competition

$$\boxed{\max_{q_1} \pi_1}$$
s.t.

- the payoffs of players:

$$\pi_1 = (120 - q_1 - q_2)q_1 - cq_1 = (120 - c)q_1 - q_1^2 - q_1 q_2$$
$$\pi_2 = (120 - q_1 - q_2)q_2 - cq_2 = (120 - c)q_2 - q_2^2 - q_1 q_2$$

- First-order conditions give us **best response functions**, also called **reaction functions**.
- The reaction function for player 1:

$$\frac{\partial \pi_1}{\partial q_1} = 120 - c - 2q_1 - q_2 \stackrel{\text{set}}{=} 0 \implies \boxed{R_1 : 120 - c - 2q_1 - q_2 = 0.}$$

$$\frac{\partial^2 \pi_1}{\partial q_1} = -2 < 0. \checkmark$$

- The reaction function for player 2:

$$\frac{\partial \pi_2}{\partial q_2} = 120 - c - 2q_2 - q_1 \stackrel{\text{set}}{=} 0 \implies \boxed{R_2 : 120 - c - 2q_2 - q_1 = 0.}$$

$$\frac{\partial^2 \pi_2}{\partial q_2} = -2 < 0. \checkmark$$

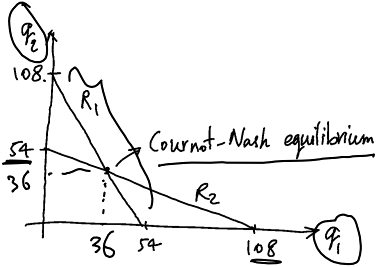
Cournot competition

- The reaction functions:

$$\frac{\partial \pi_1}{\partial q_1} = 120 - c - 2q_1 - q_2 \overset{\text{set}}{=} 0 \implies \underline{R_1 : 120 - c - 2q_1 - q_2 = 0.}$$
$$\frac{\partial \pi_2}{\partial q_2} = 120 - c - 2q_2 - q_1 \overset{\text{set}}{=} 0 \implies \underline{R_2 : 120 - c - 2q_2 - q_1 = 0.}$$

- Where the reaction functions intersect is the **Cournot-Nash equilibrium**:

$$\underline{120 - c - 2q_1 - q_2 = 120 - c - 2q_2 - q_1} \implies \boxed{q_1 = q_2,}$$
$$\underline{120 - c - 2q_1 - q_2 = 0} \implies \underbrace{120 - c - 2q_1 - q_1 = 0} \implies q_1 = q_2 = \underbrace{40 - \frac{c}{3}} = \underline{36.}$$



Stackelberg competition

Stackelberg competition

$\pi_1 = (120-12)(54) - 54^2 - \cancel{(54 \times 27)}$

$\pi_2 = (\underbrace{120-12}_{108})(27) - 27^2 - \cancel{(54 \times 27)}$

- **Stackelberg** competition, proposed in 1934 [4]:
  - continuous strategy space
  - duopoly in which two firms choose output levels (quantities) in competition with each other
  - **sequential** moves; order of game playing matters.
- every company makes q_i quantities
- actions: choosing quantities q_1 and q_2 sequentially
- marginal cost per making quantity: $c = 12$
- demand (total generated quantity): $Q = q_1 + q_2$
- profit per quantity: $p(Q) = 120 - q_1 - q_2$
- the payoffs of players:

$$\pi_1 = (120 - q_1 - q_2)q_1 - cq_1 = (120 - c)q_1 - q_1^2 - q_1 q_2$$
$$\pi_2 = (120 - q_1 - q_2)q_2 - cq_2 = (120 - c)q_2 - q_2^2 - q_1 q_2$$

- Assume player 1 moves first. Then, the player 1 is named the **Stackelberg leader** and player 2 is named the **Stackelberg follower**.

$\pi_1 = 58322 - 2916$

$\pi_2 = 2916 - 729$

2187

55406

Stackelberg competition

- Assume player 1 moves first. Then, the player 1 is named the **Stackelberg leader** and player 2 is named the **Stackelberg follower**.
- Player 1 predicts the reaction function of the player 2, as was calculated in the Cournot game:

$$R_2 : 120 - c - 2q_2 - q_1 = 0 \implies q_2 = 60 - \frac{q_1 + c}{2}.$$

It substitutes this q_2 in its payoff:

$$\pi_1 = (120 - c)q_1 - q_1^2 - q_1 q_2 = (120 - c)q_1 - q_1^2 - q_1\left(60 - \frac{q_1 + c}{2}\right).$$

- First-order optimality condition:

$$\frac{\partial \pi_1}{\partial q_1} = (120 - c) - 2q_1 - 60 + q_1 + \frac{c}{2} \overset{\text{set}}{=} 0 \implies q_1 = 60 - \frac{c}{2} = 54.$$

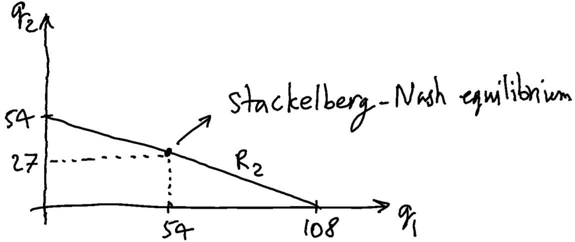
- Then, the player 2 uses its reaction function:

$$q_2 = 60 - \frac{q_1 + c}{2} = 60 - \frac{54 + c}{2} = 27.$$

- This is the **Stackelberg-Nash equilibrium**.

Stackelberg competition

$$R_2 : 120 - c - 2q_2 - q_1 = 0,$$
$$q_1 = 54.$$



Bertrand competition

Bertrand competition

simultaneously

- **Bertrand** competition, proposed in 1883 [5]:
  - continuous strategy space
  - duopoly in which two firms choose **prices, rather than output levels (quantities)**, in competition with each other
  - ~~**sequential**~~ moves; order of game playing matters.
- every company makes q_i quantities
- actions: choosing prices p_1 and p_2 from $[0, \infty)$, sequentially
- marginal cost per making quantity: $c = 12$
- the quantities are functions of prices:

$$\begin{cases} q_1 = 24 - 2p_1 + p_2, \\ q_2 = 24 - 2p_2 + p_1. \end{cases}$$

- the payoffs of players:

$q_1 p_1 - q_1 c$

$$\pi_1 = q_1(p_1 - c) = (24 - 2p_1 + p_2)(p_1 - c)$$
$$\pi_2 = q_2(p_2 - c) = (24 - 2p_2 + p_1)(p_2 - c)$$

Bertrand competition

- **Bertrand** competition:
  - the payoffs of players:

$$\pi_1 = q_1(p_1 - c) = (24 - 2p_1 + p_2)(p_1 - c)$$
$$\pi_2 = q_2(p_2 - c) = (24 - 2p_2 + p_1)(p_2 - c)$$

- First-order optimality condition:

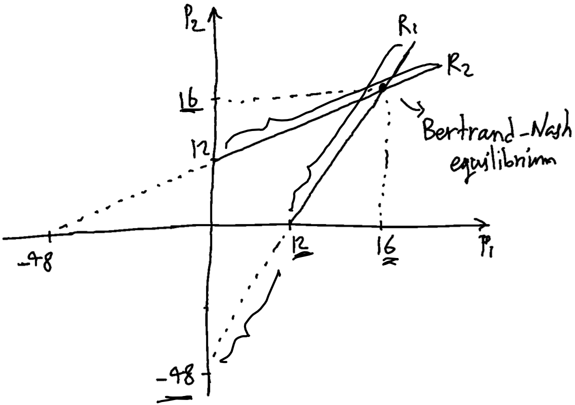
$$\frac{\partial \pi_1}{\partial p_1} = 24 - 4p_1 + p_2 + 2c \overset{\text{set}}{=} 0 \implies \boxed{R_1 : 24 - 4p_1 + p_2 + 2c = 0.}$$
$$\frac{\partial \pi_2}{\partial p_2} = 24 - 4p_2 + p_1 + 2c \overset{\text{set}}{=} 0 \implies \boxed{R_2 : 24 - 4p_2 + p_1 + 2c = 0.}$$

- Where the reaction functions intersect is the **Bertrand-Nash equilibrium**:

$$24 - 4p_1 + p_2 + 2c = 24 - 4p_2 + p_1 + 2c \implies \boxed{p_1 = p_2,}$$
$$24 - 4p_1 + p_2 + 2c = 0 \implies 24 - 4p_1 + p_1 + 2c = 0 \implies p_1 = p_2 = 8 + \frac{2c}{3} = 16.$$

Bertrand competition

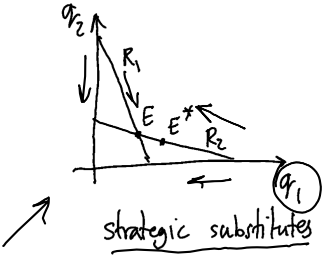
$$R_1 : 24 - 4p_1 + p_2 + 2c = 0,$$
$$R_2 : 24 - 4p_2 + p_1 + 2c = 0.$$

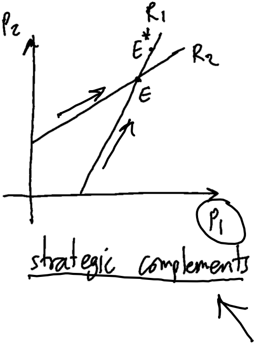


Strategic Substitutes and Complements

Strategic Substitutes and Complements

- The Cournot and Bertrand competitions are having **strategic substitutes** and **strategic complements**, respectively.
- In the Cournot competition (strategic substitutes), if a company **increases** its quantity, the other company **reduces** its quantity.
- In the Bertrand competition (strategic complements), if a company **increases** its price, the other company also **increases** its price.





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- Some slides of this slide deck are based on the following book: Eric Rasmusen, "Games and Information: An Introduction to Game Theory", 4th Edition, 2007, [6] https://www.rasmusen.org/GI/download.htm

References

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[6] E. Rasmusen, *Games and information: An introduction to game theory*. Wiley-Blackwell, 4 ed., 2007.