Game Theory: Dominance and Nash Equilibria

Adaptive and Cooperative Algorithms (ECE 457A)

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Equilibrium

Equilibrium

- **Equilibrium**: a strategy profile $\boldsymbol{s}^* = [s_1^*, s_2^*, \dots, s_n^*]^\top$ which is the best strategy for each of the n players in the game.
- **Equilibrium strategies**: the strategies selected by players maximizing their individual payoffs given the strategies of the other players.
- In game theory, we desire to find equilibrium or equilibria in games.
- Two well-known types of equilibria exist:
  - **Dominant strategy equilibrium**
  - **Nash equilibrium**

Dominant strategy equilibrium

Dominant strategy equilibrium

- We define s_{-i} to include the strategies of all players except the i-th player:

$$s_{-i} := [s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n]^\top. \tag{1}$$

- As the players are assumed to be rational, the i-th player's best response to the strategies s_{-i} chosen by the other players is the strategy s_i^* resulting in the most payoff for the i-th player:

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}), \quad \forall s_i' \neq s_i^*. \tag{2}$$

- **Dominated strategy**: a strategy of the i-th player is a dominated strategy, denoted by s_i^d, if it is strictly inferior to at least some other strategy of the i-th player regardless of what strategies the other players choose.

$$\exists s_i' : \pi_i(s_i^d, s_{-i}) < \pi_i(s_i', s_{-i}), \quad \forall s_{-i}. \tag{3}$$

- **Dominant strategy**: a strategy of the i-th player is a (strictly) dominant strategy, denoted by s_i^*, if it is strictly greater than all other strategies which the i-th player can choose regardless of what strategies the other players choose.

$$\pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}), \quad \forall s_i' \neq s_i^*, \forall s_{-i}. \tag{4}$$

Dominant strategy equilibrium

- **Weak dominant strategy**: a strategy of the *i*-th player is a weak dominant strategy, denoted by s_i^*, if it results in a higher payoff in some strategy profile and never resulting in a lower payoff. In other words, its payoff is greater than or equal to other strategies of the *i*-th player for all strategies of other players. Moreover, its payoff is strictly greater than other strategies of the *i*-th player for at least some strategies of other players [1]:

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i', s_{-i}), \quad \forall s_i' \neq s_i^*, \forall s_{-i}, \tag{5}$$

$$\exists s_{-i} : \pi_i(s_i^*, s_{-i}) > \pi_i(s_i', s_{-i}), \quad \forall s_i' \neq s_i^*. \tag{6}$$

- To summarize, a **weakly dominant strategy** is a strategy which is always at least as good as every other strategy and better than some.
- **Weakly dominant strategy equilibrium**: the strategy profile found by deleting all the weakly dominated strategies of each player.
- **Strictly (Strongly) dominant strategy equilibrium**: the strategy profile found by deleting all the strictly dominated strategies of each player.

Iterated-dominance equilibrium

- **Iterated-dominance equilibrium**:
  - One way to find the dominant strategy equilibrium is the iterated-dominance equilibrium.
  - For this, we delete a strictly/weakly dominated strategy from the strategy set of one of the players. This reduces the game matrix to a smaller matrix with less number of cases. We perform this deletion repeatedly. If we can end up with on cell finally, that cell is the strictly/weakly dominant strategy equilibrium.

Iterated-dominance equilibrium: Example

1

	b_1	b_2	b_3
a_1	13,3	1,4	7,3
a_2	4,1	3,3	6,2
a_3	-1,9	2,8	8,-1

2

	b_1	b_2	b_3
a_1	13,3	1,4	7,3
a_2	4,1	3,3	6,2
a_3	-1,9	2,8	8,-1

$>$

3

	b_1	b_2
a_1	13,3	1,4
a_2	4,1	3,3
a_3	-1,9	2,8

V

4

	b_1	b_2	b_3
a_1	13,3	1,4	7,3
a_2	4,1	3,3	6,2
a_3	-1,9	2,8	8,-1

$\not\succ$

5

	b_1	b_2
a_1	13,3	1,4
a_2	4,1	3,3

$<$

6

	b_2
a_1	1,4
a_2	3,3

$\wedge$

7

	b_2
a_2	3,3

Nash Equilibrium

Nash Equilibrium

- Nash equilibrium was proposed by **John Nash** during years 1949 to 1953 [2, 3, 4, 5]. See his Google Scholar:
  `https://scholar.google.com/citations?user=mYuYWJkAAAAJ&hl=en&oi=sra`
- **Nash equilibrium**: the strategy profile s^* is a Nash equilibrium if no player has incentive to deviate from its strategy given that the other players do not deviate [1, 6]:

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \quad \forall s_i'. \tag{7}$$

- Comparing this equation with Eq. (4) shows that the Nash equilibrium does not have $\forall s_{-i}$.
- In other words, in the Nash equilibrium, all players are happy with their situation and do not wish to deviate from the equilibrium.
- **Strict (Strong) Nash equilibrium**:

$$\pi_i(s_i^*, s_{-i}^*) > \pi_i(s_i', s_{-i}^*), \quad \forall s_i'. \tag{8}$$

- **Weak Nash equilibrium**:

$$\pi_i(s_i^*, s_{-i}^*) \geq \pi_i(s_i', s_{-i}^*), \quad \forall s_i'. \tag{9}$$

- **Every dominant strategy equilibrium is a Nash equilibrium but not vice versa.**

Nash Equilibrium

- A way to find the Nash equilibrium in a game is to start from one of the cells in the game matrix and move (deviate) to an adjacent cell if the payoff of the adjacent cell is strictly/weakly greater than that cell. We do this for all cells and players and show the movements by arrows between the cells. The cell(s) where the arrows converge to are the strict/weak Nash equilibria.

Understanding Nash Equilibrium by A Movie Scene

- The bar scene in the movie “A Beautiful Mind” about John Nash.

Nash Equilibrium: Example

	b_1	b_2	b_3
a_1	13, 3	1, 4	7, 3
a_2	4, 1	3, 3	6, 2
a_3	-1, 9	2, 8	8, -1

strict Nash:

(a_2, b_2)

$\pi: (3, 3)$

	b_1	b_2
a_1	0, 0	-10, 0
a_2	0, -10	-8, -8

strict Nash: $\{(a_2, b_2)$, $\pi: (-8, -8)$

weak Nash: $\{(a_1, b_1)$, $\pi: (0, 0)$

Examples for dominant and Nash Equilibria

Equilibria for the Prisoner's Dilemma

prisoner 2

prisoner 1	cooperate	defect
cooperate	−1, −1	−3, 0
defect	0, −3	−2, −2

prisoner 2

prisoner 1	cooperate	defect
cooperate	−1, −1	−3, 0
defect	0, −3	−2, −2

prisoner 2

prisoner 1	cooperate	defect
cooperate	−1, −1	−3, 0
defect	0, −3	−2, −2

	defect
cooperate	−3, 0
defect	−2, −2

	defect
defect	−2, −2

Equilibria for the Game of Chicken

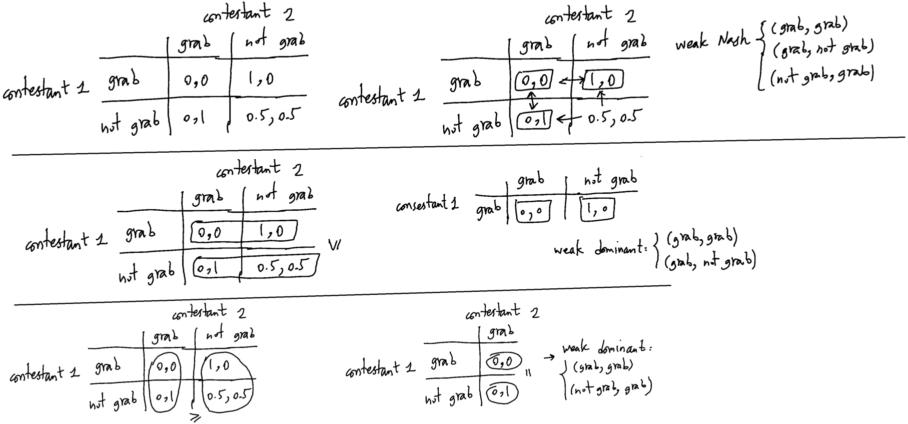
		continue	swerve
teenager 1	continue	−3, −3	2, 0
	swerve	0, 2	1, 1

		continue	swerve
teenager 1	continue	−3, −3 →	[2, 0]
	swerve	↓ [0, 2]	← 1, 1 ↑

		continue	swerve
teenager 1	continue	−3, −3	2, 0
	swerve	0, 2	1, 1

no dominant strategy

Equilibria for Grab the Dollar



Acknowledgment

- Some slides of this slide deck are inspired by teachings of Prof. Stanko Dimitrov at the University of Waterloo, Department of Management Science and Engineering.
- Some slides of this slide deck are based on the following book: Eric Rasmusen, "Games and Information: An Introduction to Game Theory", 4th Edition, 2007, [1]
  `https://www.rasmusen.org/GI/download.htm`
- A good lecture series on YouTube, by William Spaniel, about fundamentals of game theory (named "Game Theory 101: Strategic Form Games"):
  `https://www.youtube.com/playlist?list=PL7F0C4C7A4C910AF5`
- An important scholar in the area of game theory: Martin J. Osborne, who used to be a professor at the University of Toronto.
  Google Scholar:
  `https://scholar.google.com/citations?user=lx-4Hd8AAAAJ&hl=en&oi=sra`

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