Fuzzy Sets and Fuzzy Logic

Adaptive and Cooperative Algorithms (ECE 457A)

ECE, MME, and MSCI Departments,
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Introduction

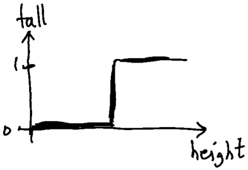
Introduction

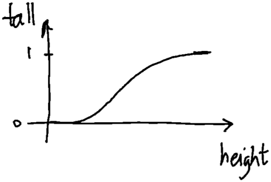
- Crisp binary (boolean) logic was proposed by George Boole (1815-1864).
- The truth table of binary logic was proposed by the famous logician-philosopher, Ludwig Wittgenstein (1889-1951).
- Fuzzy logic was proposed by Lotfi Aliasker Zadeh (1921-2017) in 1960's.
- How are you?
  - You can say: I am good or I am bad.
  - But what if you are 70% good, i.e., feeling 30% bad.
- Fuzzy logic extends or generalizes binary logic (crisp logic) from two levels (0 and 1) to a continuous range $[0, 1]$.
- In other words, it generalizes the binary logic to many-valued logic.
- So, in fuzzy logic, we have partial true and partial false rather than merely true and false.
- This imitates the human-like approximate reasoning.
- Many real-world quantities are subjective, approximate, and qualitative.

Fuzzy sets

Fuzzy sets

- Fuzzy sets are the building blocks and corner stones of fuzzy logic.
- In a crisp set, an element either is a member of the set or not. In a fuzzy set, an element can have partial membership.
- Example: the set of tall people

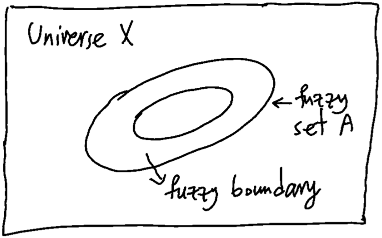




- The membership to a fuzzy set is a gray zone. For instance, in this example, there is not a crisp cut-off of height for being tall.

Fuzzy sets

- Let X be a set that contains every set of interest in the context of a given class of problems (e.g., set of all humans). Then, X is called the **universe of discourse** (or simply the universe), whose elements are denoted by x.
- A fuzzy set A in the universe of discourse X can be represented by a **Venn diagram**.

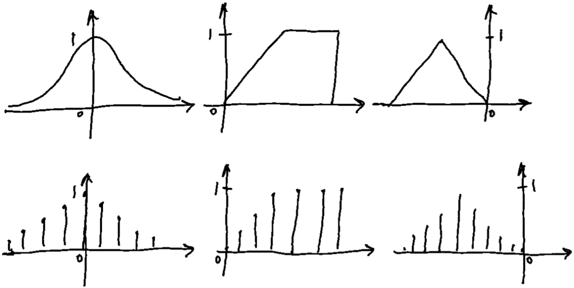


Membership function

- every element has a degree (or grade) of membership in a fuzzy set.
- This membership can be denoted by a membership function whose domain is the domain of values that the element can have and its range is $[0, 1]$.
- Let the membership function of x in the fuzzy set A be denoted by $\mu_A(x)$. Then:

$$\mu_A(x) : \mathcal{D}(x) \to [0, 1], \quad \mu_A : x \mapsto \mu_A(x). \tag{1}$$

- Some example membership functions:



Membership function

- Every element x has some membership in the fuzzy set A. Therefore, the fuzzy set A can be represented as a set of ordered pairs:

$$A = \{ (x, \mu_A(x)) \mid x \in X, \mu_A(x) \in [0,1] \}. \tag{2}$$

- Note that the membership shows the grade of **possibility** and **not probability** because it is in range $[0, 1]$ but the summation of possibility of values is not one necessarily.
- A crisp set is a special case of a fuzzy set where the membership function can take only two values 0 and 1.

Symbolic representation

- If the universe of discourse is discrete with elements $\{x_i\}_{i=1}^n$, the fuzzy set A can be represented symbolically with symbolic summation:

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}. \tag{3}$$

- If the universe of universe is continuous with elements $x \in X$, the fuzzy set A can be represented symbolically with symbolic integration:

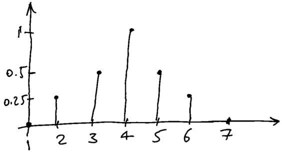
$$A = \int_{x \in X} \frac{\mu_A(x)}{x}. \tag{4}$$

- Note that the notations $\sum$ and $\int$ are not the operators for summation and integration but they are merely symbolic notations.

Symbolic representation

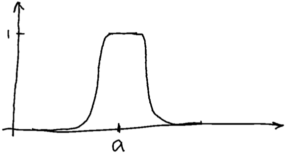
- Examples:
  - discrete fuzzy set for representing the digit four:

$$A = 0.25/2 + 0.5/3 + 1/4 + 0.5/5 + 0.25/6.$$



  - continuous fuzzy set (called fuzzy relation) for representing the value a:

$$\mu_A(x) = \frac{1}{1 + (x - a)^{10}}, \quad A = \int_x \frac{\frac{1}{1+(x-a)^{10}}}{x}$$



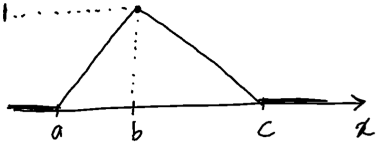
Well-known membership functions

Well-known membership functions

- Triangular membership function:

$$\mu_A(x; a, b, c) := \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } x \geq c \end{cases} = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right), \qquad (5)$$

where $a < b < c$.

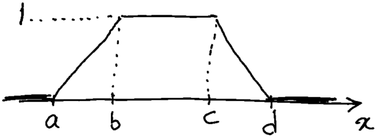


Well-known membership functions

- Trapezoidal membership function:

$$\mu_A(x; a, b, c, d) := \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } x \geq d \end{cases} = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right), \quad (6)$$

where $a < b \leq c < d$.



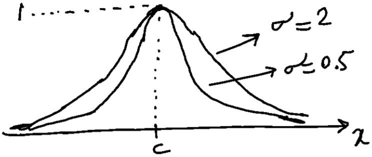
Well-known membership functions

- Gaussian membership function:

$$\mu_A(x; c, \sigma^2) := e^{\frac{-(x-c)^2}{2\sigma^2}} \tag{7}$$

  where c is the mean and σ^2 is the variance of the Gaussian function.

- Note that the integral of this function is not one (it is not a probability density function) which is fine for fuzzy membership function. The important thing is that at $x = c$, it becomes 1.

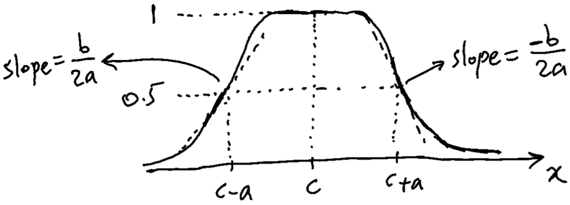


Well-known membership functions

- Generalized bell (also called bell) membership function:

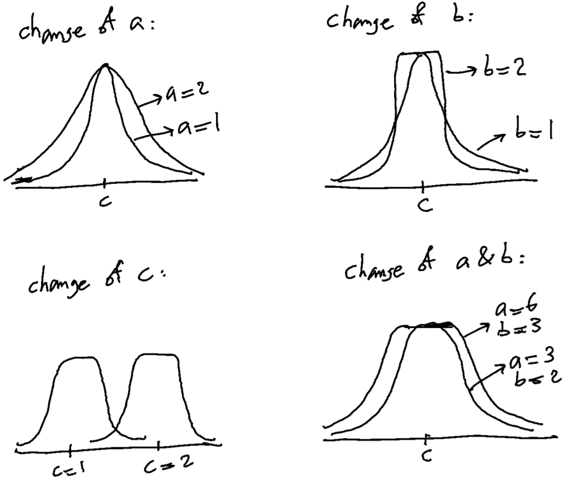
$$\mu_A(x; a, b, c) := \frac{1}{1 + |\frac{x-c}{a}|^{2b}} \tag{8}$$

where c is the mean of the bell and both a and b control the slope of bell and heaviness of its tail. The slope of the bell is $\pm b/2a$. The parameter a also controls how wide the top of the bell is.



Well-known membership functions

- Generalized bell (also called bell) membership function:



Comparing well-known membership functions

- Each of the membership functions has its own pros and cons.
- The triangular and trapezoidal functions have linear parts so they are inexpensive computationally. However, they are non-smooth and non-differentiable so they are not suitable for gradient-based optimization.
- The Gaussian and bell functions are nonlinear so they are expensive computationally. However, they are smooth and differentiable so they are suitable for gradient-based optimization.

Main fuzzy logic operations

Main fuzzy logic operations

- There exist operations on fuzzy sets similar to the operations on crisp sets.
- A fuzzy logic operation is an operation which is applied to one or multiple membership functions and outputs one membership function:

$$f : [0,1] \times \cdots \times [0,1] \to [0,1]. \tag{9}$$

- Three most important operations are:
  - T-norm (intersection)
  - S-norm (union)
  - Complement

T-norm

- The intersection of two fuzzy sets is performed by the T-norm operator:

$$\mu_{A\cap B}(x) = T\left(\mu_A(x), \mu_B(x)\right) = \mu_A(x) T \mu_B(x). \tag{10}$$

- The properties of the T-norm operator:
  - non-decreasing: if $a \le b$ and $c \le d$, then $aTc \le bTd$.
  - commutativity: $aTb = bTa$
  - associativity: $(aTb)Tc = aT(bTc)$
  - boundary conditions: $aT0 = 0$ and $aT1 = a$
- Well-known T-norm operators:
  - min: $T(a, b) = \min(a, b) = a \wedge b$
  - algebraic product: $T(a, b) = a \times b$
  - bounded product: $T(a, b) = 0 \vee (a + b - 1)$
  - basic product: $T(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$
  - general form 1: $T(a, b) = 1 - \min\Big(1, \left((1-a)^p + (1-b)^p\right)^{1/p}\Big), \quad p \ge 1$
  - general form 2: $T(a, b) = \max\big(0, (\lambda + 1)(a + b - 1) - \lambda ab\big), \quad \lambda \ge -1$
- The most common T-norm is the min operator.

S-norm

- The union of two fuzzy sets is performed by the S-norm operator (also called T-conorm operator):

$$\mu_{A\cup B}(x) = S\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) S \mu_B(x). \tag{11}$$

- The properties of the S-norm operator:
  - non-decreasing: if $a \leq b$ and $c \leq d$, then $aSc \leq bSd$.
  - commutativity: $aSb = bSa$
  - associativity: $(aSb)Sc = aS(bSc)$
  - boundary conditions: $aS0 = a$ and $aS1 = 1$
- Well-known S-norm operators:
  - max: $S(a,b) = \max(a,b) = a \vee b$
  - algebraic sum: $S(a,b) = a + b - ab$
  - bounded product: $S(a,b) = 1 \wedge (a+b)$
  - basic product: $S(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{if } a, b < 1 \end{cases}$
  - general form 1: $S(a,b) = \min\big(1, (a^p + b^p)^{1/p}\big), \quad p \geq 1$
  - general form 2: $S(a,b) = \min\big(1, a + b + \lambda ab\big), \quad \lambda \geq -1$
- The most common S-norm is the max operator.

Complement

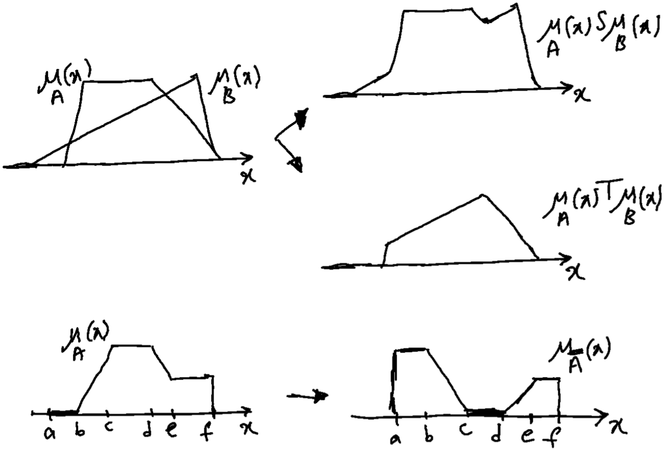
- The complement of a fuzzy set is performed by the complement operator:

$$\mu_{\bar{A}}(x) = Co\big(\mu_A(x)\big), \tag{12}$$

  where $\bar{A}$ denotes the complement of the set A.
- The properties of the complement operator:
  - non-increasing: if $a \leq b$, then $Co(a) \geq Co(b)$.
  - involutive: $Co\big(Co(a)\big) = a$
  - boundary conditions: $Co(0) = 1$ and $Co(1) = 0$
- Well-known complement operators:
  - regular complement: $Co(a) = 1 - a$
  - Sugeno's complement: $Co(a) = \frac{1-a}{1+pa}, \quad p \in (-1, \infty)$
  - Yager's complement: $Co(a) = (1 - a^p)^{1/p}, \quad p \in (0, \infty)$
- The most common complement is the regular complement operator.

Main fuzzy logic operations

- Examples for fuzzy logic operations:



Operations on single fuzzy sets

Operations on single fuzzy sets

- There exist some other operations for fuzzy sets:
  - height (modal grade)
  - support set
  - α-cut
  - measures of fuzziness
  - set dilation and contraction

Height (modal grade)

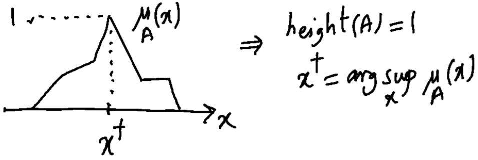
- The height or modal grade of a fuzzy set A is the maximum of its membership function:

$$\text{height}(A) = \sup_{x \in X} \mu_A(x). \tag{13}$$

- The value of x with the maximum membership is defined as the modal point or the modal element value:

$$x^\dagger = \arg\sup_{x \in X} \mu_A(x), \tag{14}$$

$$\mu_A(x^\dagger) = \text{height}(A). \tag{15}$$



Support set and α-cut

- The support set of a fuzzy set is a crisp set containing all the elements in the universe whose membership grades are strictly greater than zero:

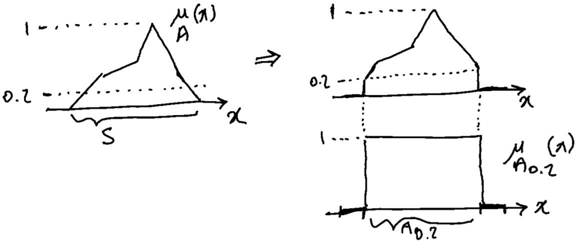
$$S := \{x \in X \mid \mu_A(x) > 0\}. \tag{16}$$

- The α-cut of a fuzzy set A is the crisp set denoted by A_α formed by the elements of A whose membership function grades are greater than or equal to a specified threshold value $\alpha \in [0, 1]$:

$$A_\alpha := \{x \in X \mid \mu_A(x) \geq \alpha\}. \tag{17}$$

Therefore, we define:

$$\mu_{A_\alpha}(x) = \begin{cases} 1 & \text{if } \mu_A(x) \geq \alpha \\ 0 & \text{Otherwise.} \end{cases} \tag{18}$$



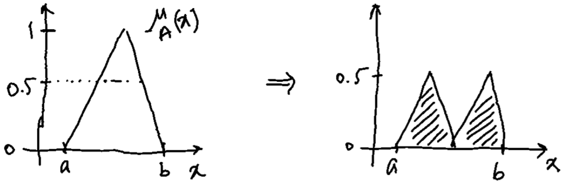
Measures of fuzziness

- Every membership function for a fuzzy set has some fuzziness. There exist various measures for fuzziness of a set [1, 2, 3]. The larger the fuzziness measurement, the more ambiguous (dilated) its membership function is. Some of them are:
  - closeness to grade 0.5:

$$\text{fuzziness} = \int_{x \in S} f(x)\, dx, \tag{19}$$

$$f(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \leq 0.5 \\ 1 - \mu_A(x) & \text{Otherwise,} \end{cases} \tag{20}$$

where the integral notation is for integrating and not the symbolic representation.

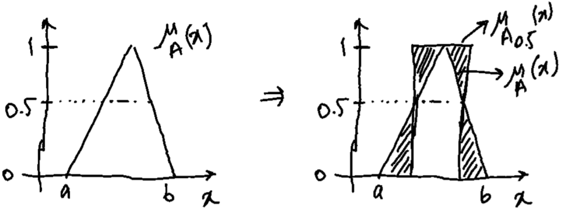


Assuming that $x \in [a, b]$.

Measures of fuzziness

- Some of them are:
  - distance from 0.5-cut [4]:

$$\text{fuzziness} = \int_{x \in X} \left| \mu_A(x) - \mu_{A_{0.5}}(x) \right| dx. \tag{21}$$



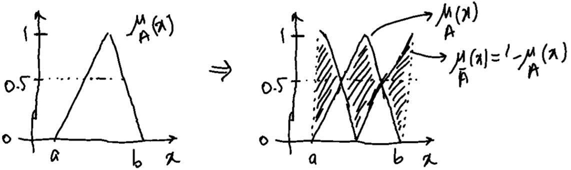
Assuming that $x \in [a, b]$.

Measures of fuzziness

- Some of them are:
  - inverse of distance from the complement:

$$\text{fuzziness} = \int_{x \in X} \left| \mu_A(x) - \mu_{\bar{A}}(x) \right| dx, \tag{22}$$

where $\bar{A}$ is the complement of A and $|.|$ is the absolute value function.



Assuming that $x \in [a, b]$.

  - Another variant of inverse of distance from the complement [5]:

$$\text{fuzziness} = \int_{x \in X} \mu_A(x)\mu_{\bar{A}}(x)\, dx, \tag{23}$$

Set dilation and contraction

- The k-th dilation of a fuzzy set A is defined as:

$$\mu_{A'}(x) = \mu_A^{1/k}(x). \tag{24}$$

- The k-th contraction of a fuzzy set A is defined as:

$$\mu_{A'}(x) = \mu_A^{k}(x). \tag{25}$$

- In notations, the dilation and contraction operators are defined as:

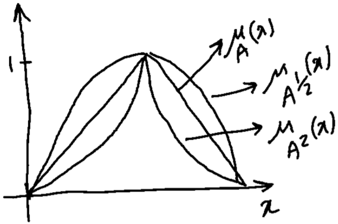
$$\mathrm{dil}(A) = A^{1/k} = \int \frac{\mu_A^{1/k}(x)}{x}, \tag{26}$$

$$\mathrm{con}(A) = A^{k} = \int \frac{\mu_A^{k}(x)}{x}. \tag{27}$$

- As $\mu_A(x) \in [0, 1]$, $\mu_A^{1/k}(x)$ and $\mu_A^{k}(x)$ dilates and contracts the membership function, respectively.
- Let $\mu_A(x)$ be the membership function for a quality. Therefore, on the one hand, dilation refers to "more or less" or "somewhat" of that quality. On the other hand, contraction refers to "very" or "too" for that quality.

Set dilation and contraction

- The illustration of dilation and contraction are as follows.

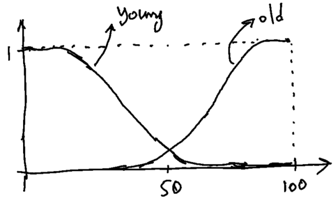


Set dilation and contraction

- Example for fuzzy sets of age (credit of this example is for Prof. Karray):

$$\mu_{young}(x) = \frac{1}{1 + \left| \frac{x}{20} \right|^4}$$

$$\mu_{old}(x) = \frac{1}{1 + \left| \frac{x-100}{30} \right|^6}$$



Set dilation and contraction

* more or less old = 2nd dilation of old

$$\mu_{\text{more or less old}}(x) = \sqrt{\frac{1}{1+\left|\frac{x-100}{30}\right|^6}}$$

* not young and not old = $\overline{\text{young}} \wedge \overline{\text{old}}$

$$\mu_{\overline{\text{young}} \wedge \overline{\text{old}}}(x) = \left(1 - \frac{1}{1+\left|\frac{x}{20}\right|^4}\right) \wedge \left(1 - \frac{1}{1+\left|\frac{x-100}{30}\right|^6}\right)$$

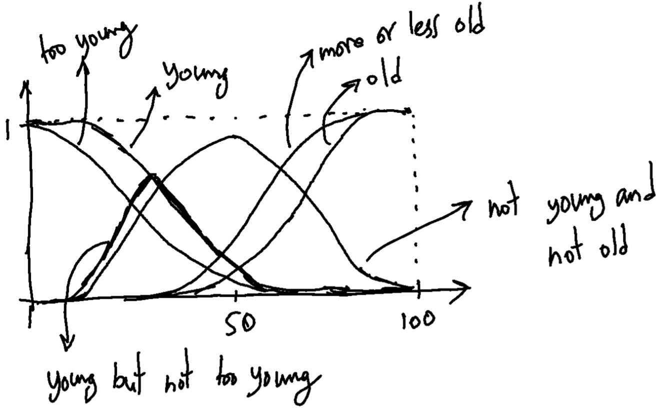
* too young = 2nd contraction of young

$$\mu_{\text{too young}}(x) = \left(\frac{1}{1+\left|\frac{x}{20}\right|^4}\right)^2$$

* young but not too young = young $\wedge$ $\overline{\text{too young}}$

$$\mu_{\text{young but not too young}} = \left(\frac{1}{1+\left|\frac{x}{20}\right|^4}\right) \wedge \left(1 - \left(\frac{1}{1+\left|\frac{x}{20}\right|^4}\right)^2\right)$$

Set dilation and contraction



Operations for relation of fuzzy sets

Operations for relation of fuzzy sets

- There exist several operations for the relation of fuzzy sets similar to the relation of crisp sets.
- Well-known operations for the relation of fuzzy sets are:
  - set inclusion
  - set equality
  - implication (if-then)
  - extension principle
  - projection
  - cylindrical extension

Set inclusion

- In crisp sets, a set either is a subset of another set or not. However, a fuzzy set can be partially subset of another fuzzy set.
- The grade of inclusion for the partial set inclusion of a fuzzy set A in another fuzzy set B is defined as:

$$\mu_{A\subset B}(x) = \begin{cases} 1 & \text{if } \mu_A(x) < \mu_B(x) \\ \mu_A(x)\, T\, \mu_B(x) & \text{Otherwise.} \end{cases} \quad (28)$$

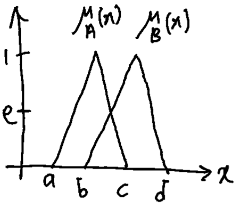
$$\mu_{A\subseteq B}(x) = \begin{cases} 1 & \text{if } \mu_A(x) \leq \mu_B(x) \\ \mu_A(x)\, T\, \mu_B(x) & \text{Otherwise.} \end{cases} \quad (29)$$

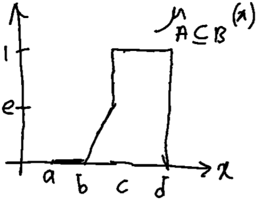
- A fuzzy set A is completely included in another fuzzy set B if:

$$A \subset B \iff \mu_A(x) < \mu_B(x), \quad \forall x \in X, \quad (30)$$

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \quad \forall x \in X. \quad (31)$$

- If $A \subset B$, A is said to be the proper subset of B.





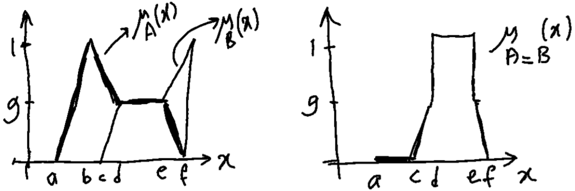
Set equality

- The equality of two fuzzy sets is a special case of set inclusion.
- The grade of equality for the partial set equality of a fuzzy set A to another fuzzy set B is defined as:

$$\mu_{A=B}(x) = \begin{cases} 1 & \text{if } \mu_A(x) = \mu_B(x) \\ \mu_A(x)\, T\, \mu_B(x) & \text{Otherwise.} \end{cases} \tag{32}$$

- A fuzzy set A is completely equal to another fuzzy set B if:

$$A = B \iff \mu_A(x) = \mu_B(x), \quad \forall x \in X. \tag{33}$$



Implication (if-then)

- Consider two fuzzy sets which may be in the same universe or in two different universes, i.e., $A \in X$, $B \in Y$.
- The fuzzy implication $A \rightarrow B$ is a fuzzy relation in the Cartesian product $X \times Y$.
- The fuzzy implication $A \rightarrow B$ means "if A is (partially) true, then it implies that B is (partially) true.
- Note that according to logic, $A \rightarrow B$ is equivalent to having $\bar{B} \not\rightarrow \bar{A}$, where $\bar{A}$ is the complement of the set A. It means that "if B is (partially) false, then it implies that A is (partially) false.
- Examples:
  - example for $A \rightarrow B$: if it (partially) rains, then the ground becomes (partially) wet.
  - example for $\bar{B} \not\rightarrow \bar{A}$: the ground is not (partially) wet. Therefore, it must have not (partially) rained.
- The implication $A \rightarrow B$ is also referred to as the **rule of inference**. The rule of inference is also called the **modus ponens** in binary logic.
  - The implication $A \rightarrow B$ in the binary logic means as follows: The rule says if x is A, then y is B. Now, x is A; therefore, y is B.
  - The implication $A \rightarrow B$ in the fuzzy logic means as follows: The rule says if x is A, then y is B. Now, x is A'; therefore, y is B', where A' and B' can be far from or close to A and B, respectively.

Implication (if-then)

- There exist different implication operators in fuzzy logic:
  - Larsen implication:

$$\mu_{A \to B}(x, y) = \mu_A(x)\mu_B(y), \quad \forall (x, y) \in X \times Y. \tag{34}$$

  - Mamdani implication:

$$\mu_{A \to B}(x, y) = \min\big(\mu_A(x), \mu_B(y)\big), \quad \forall (x, y) \in X \times Y. \tag{35}$$

  - Zadeh implication:

$$\mu_{A \to B}(x, y) = \max\Big(\min\big(\mu_A(x), \mu_B(y)\big), 1 - \mu_A(x)\Big), \quad \forall (x, y) \in X \times Y. \tag{36}$$

  - Dienes-Rascher implication:

$$\mu_{A \to B}(x, y) = \max\big(1 - \mu_A(x), \mu_B(y)\big), \quad \forall (x, y) \in X \times Y. \tag{37}$$

  - Lukasiewicz implication:

$$\mu_{A \to B}(x, y) = \min\big(1, 1 - \mu_A(x) + \mu_B(y)\big), \quad \forall (x, y) \in X \times Y. \tag{38}$$

- The most common implication operator is Mamdani implication.

Extension principle

- Consider a map from universe X to universe Y, i.e., $f : X \to Y$
- Let A be a set in universe X

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n}. \tag{39}$$

- Let $B = f(A)$ be a set in universe Y:

$$B = f(A) = \frac{\mu_A(x_1)}{y_1} + \cdots + \frac{\mu_A(x_n)}{y_n}. \tag{40}$$

- If the map $f(.)$ is a one-to-one map, then:

$$y_1 = f(x_1), \ldots, y_n = f(x_n) \implies \mu_B(y_i) = \mu_A(x_i). \tag{41}$$

- If the map $f(.)$ is a many-to-one map, then the S-norm of their membership functions is used:

$$\exists x_1 \neq x_2 : y_1 = y_2 = f(x_1) = f(x_2) \implies \mu_B(y_1) = \mu_B(y_2) = \max\big(\mu_A(x_1), \mu_A(x_2)\big). \tag{42}$$

Extension principle

- Numerical example for the extension principle in discrete membership functions:

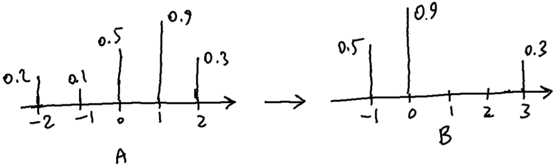
$$A = \frac{0.2}{-2} + \frac{0.1}{-1} + \frac{0.5}{0} + \frac{0.9}{1} + \frac{0.3}{2}$$

$$f : x \mapsto x^2 - 1$$

$$B = \frac{0.2}{3} + \frac{0.1}{0} + \frac{0.5}{-1} + \frac{0.9}{0} + \frac{0.3}{3}$$

$$= \frac{(0.2 \vee 0.3)}{3} + \frac{(0.1 \vee 0.9)}{0} + \frac{0.5}{-1}$$

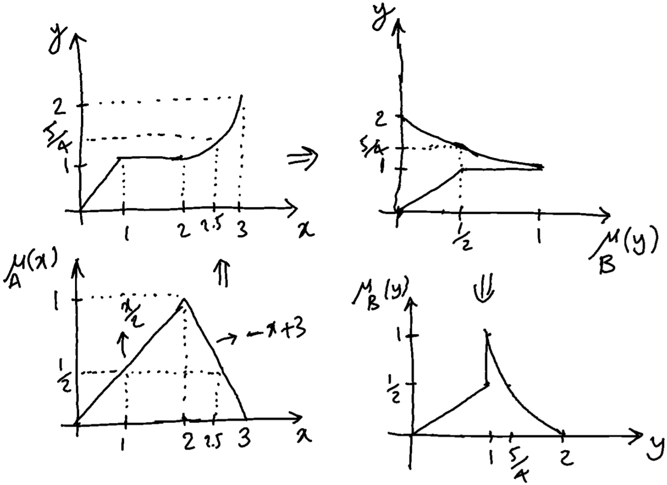
$$= \frac{0.3}{3} + \frac{0.9}{0} + \frac{0.5}{-1}$$



Extension principle

- Visual example for the extension principle in continuous membership functions:

$$x \in [0,3], \quad y = f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 1 & \text{if } 1 \leq x \leq 2 \\ (x-2)^2 + 1 & \text{if } x \geq 2. \end{cases}$$



Projection

- Consider the Cartesian product $X \times Y$ with the set:

$$R = \int_{X \times Y} \frac{\mu_R(x,y)}{(x,y)},$$

where this set can be the implication or relation set between sets X and Y.

- The projection of this relation set onto the set X is:

$$R_1 = \int_X \frac{\mu_{R_1}(x)}{x}, \quad \mu_{R_1}(x) = \bigvee_y \mu_R(x,y), \tag{43}$$

where $\bigvee_y$ is the max S-norm over y.

- The projection of this relation set onto the set Y is:

$$R_2 = \int_Y \frac{\mu_{R_2}(y)}{y}, \quad \mu_{R_2}(y) = \bigvee_x \mu_R(x,y), \tag{44}$$

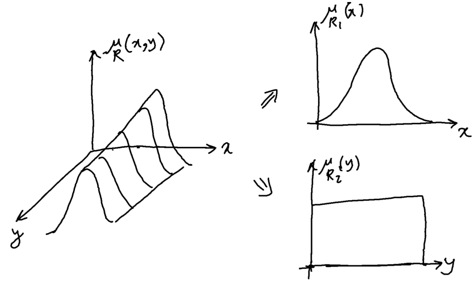
where $\bigvee_x$ is the max S-norm over x.

- The total projection of this relation set onto the sets X and Y is:

$$\mu_{R_T}(x,y) = \bigvee_x \bigvee_y \mu_R(x,y). \tag{45}$$

Projection

- Visual example for projection:



- Numerical example for projection: consider the following Cartesian product of X and Y:

$$R = \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.8 \\ 0.2 & 0.4 & 0.8 & 0.9 \\ 0.5 & 0.9 & 1.0 & 0.8 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 0.8 \\ 0.9 \\ 1.0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.5 \\ 0.9 \\ 1.0 \\ 0.9 \end{bmatrix} = [0.5, 0.9, 1.0, 0.9]^\top$$

$$R_T = 1$$

Cylindrical extension

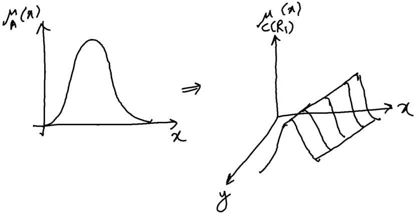
- Consider the Cartesian product of n fuzzy sets $X_1, \ldots, X_n$.
- The cylindrical extension of the fuzzy set A over this Cartesian product is defined as:

$$C(A) := \int_{X_1 \times \cdots \times X_n} \frac{\mu_A(X_1, \ldots, X_n)}{(X_1, \ldots, X_n)}. \tag{46}$$

- Example: consider the following Cartesian product of X and Y:

$$R = \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.8 \\ 0.2 & 0.4 & 0.8 & 0.9 \\ 0.5 & 0.9 & 1.0 & 0.8 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.8 \\ 0.9 \\ 1.0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0.5 \\ 0.9 \\ 1.0 \\ 0.9 \end{bmatrix} = [0.5, 0.9, 1.0, 0.9]^\top$$

$$C(R_1) = \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.9 & 0.9 \\ 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}, \quad C(R_2) = \begin{bmatrix} 0.5 & 0.9 & 1.0 & 0.9 \\ 0.5 & 0.9 & 1.0 & 0.9 \\ 0.5 & 0.9 & 1.0 & 0.9 \end{bmatrix}$$



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  - George Klir, Bo Yuan, “Fuzzy Sets and Fuzzy Logic: Theory and Applications”, 1995 [6]
  - Lotfi A. Zadeh, George J Klir, Bo Yuan, “Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers”, 1996 [7]
  - Fakhreddine O Karray, Clarence W De Silva, “Soft computing and intelligent systems design: theory, tools, and applications”, 2004 [8]
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- Important journals in the area of fuzzy logic:
  - IEEE Transactions on Fuzzy Systems
  - Fuzzy Sets and Systems, Elsevier
  - Knowledge-Based Systems, Elsevier
  - Applied Soft Computing, Elsevier
  - Soft Computing, Springer
  - Fuzzy Optimization and Decision Making, Springer

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