Fuzzy Inference System

Adaptive and Cooperative Algorithms (ECE 457A)

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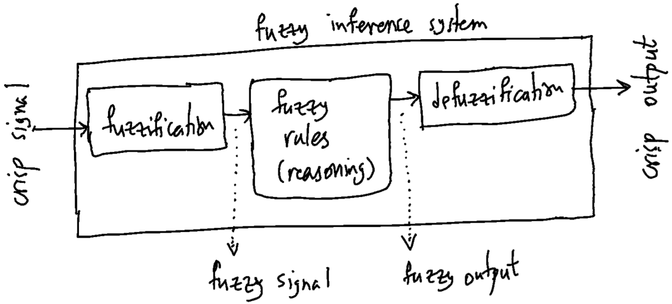
Fuzzy inference system

Fuzzy inference system

- A Fuzzy Inference System (FIS) has three main parts:
  - fuzzification
  - fuzzy rules (fuzzu reasoning)
  - defuzzification



Fuzzification

Fuzzification in fuzzy system

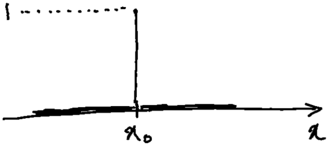
- The input of fuzzy rules should be fuzzy membership function values.
- This is while the signals in the real-world are crisp and not fuzzy. Therefore, the crisp signals should be converted to fuzzy membership functions.
- Fuzzification refers to the representation of a crisp value by a membership function.
- This is also justified because a measured signal may not be known to be 100% accurate. The fuzzy membership values account for the possible noise in the measured crisp signals.
- There are several methods for fuzzification. The well-known fuzzification methods are:
  - singleton method
  - triangular function method
  - Gaussian function method

Fuzzification in fuzzy system

- The singleton method for fuzzification makes the following membership function for the crisp value x_0 as follows:

$$\mu(x) = \delta(x - x_0) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0, \end{cases} \quad (1)$$

where $\delta(.)$ is the Kronecker delta. This fuzzification method treats the crisp value as a fuzzy value without any fuzziness.

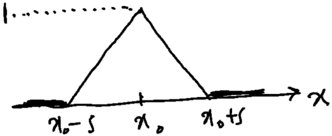


Fuzzification in fuzzy system

- The triangular function method for fuzzification makes the following membership function for the crisp value x_0 as follows:

$$\mu(x) = \begin{cases} 1 - \frac{|x-x_0|}{s} & \text{if } |x - x_0| \leq s \\ 0 & \text{if } |x - x_0| > s, \end{cases} \tag{2}$$

where $|.|$ denotes the absolute value function and s is the length of support set, called the base length.

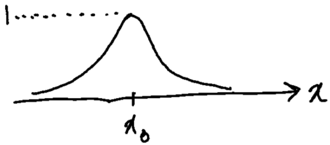


Fuzzification in fuzzy system

- The Gaussian function method for fuzzification makes the following membership function for the crisp value x_0 as follows:

$$\mu(x) = \exp\left(\left(\frac{x - x_0}{\sigma}\right)^2\right), \tag{3}$$

where σ is the standard deviation of the Gaussian function.



Defuzzification

Defuzzification in fuzzy system

- The output of fuzzy rules is a fuzzy value.
- This is while the signals in the real-world are crisp and not fuzzy. Therefore, the fuzzy output of fuzzy inference should be converted to a crisp value.
- The defuzzified value can be used as an actuator signal for the controllers, for example.
- Defuzzification refers to the representation of a membership function as a crisp value.
- There are several methods for defuzzification. The well-known defuzzification methods are:
  - centroid method
  - mean of maxima method
  - smallest of maximum
  - largest of maximum
  - bisector of area
  - threshold method

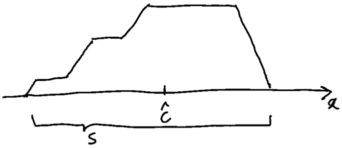
Defuzzification in fuzzy system

- The **centroid method** for defuzzification makes the following crisp value $\widehat{c}$ from the continuous and discrete membership function $\mu(x)$ as follows:

$$\widehat{c} = \frac{\int_{x \in S} x \, \mu(x) dx}{\int_{x \in S} \mu(x) dx}, \quad (4)$$

$$\widehat{c} = \frac{\sum_{x_i \in S} x_i \, \mu(x_i)}{\sum_{x_i \in S} \mu(x_i)}, \quad (5)$$

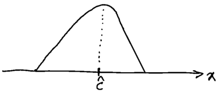
respectively, where the integral sign is for integration and it is not a symbolic representation.



Defuzzification in fuzzy system

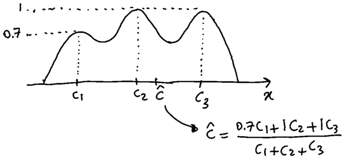
- If the membership function is **unimodal**, the **Mean Of Maxima (MOM) method** for defuzzification makes the following crisp value $\widehat{c}$ from the membership function $\mu(x)$ as follows:

$$\widehat{c} = \arg\max_{x} \mu(x). \tag{6}$$



- Assume the membership function is **multi-modal**, meaning that it has multiple local maxima where the maximum membership values may or may not be equal to each other. Let m denote the number of modes of the membership function. Let the membership values of the local modes be $\mu_1, \ldots, \mu_m$ and let the x's where the local modes exist be $c_1, \ldots, c_m$. The defuzzified value $\widehat{c}$ is the weighted average of values at the modes of the membership function:

$$\widehat{c} = \frac{\sum_{i=1}^{m} \mu_i c_i}{\sum_{i=1}^{m} c_i}. \tag{7}$$



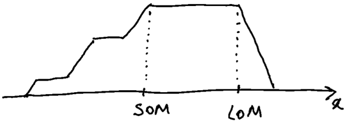
Defuzzification in fuzzy system

- The **Smallest Of Maximum (SOM) method** and the **Largest Of Maximum (LOM) method** for defuzzification make the following crisp value $\widehat{c}$ from the membership function $\mu(x)$ as follows:

$$\widehat{c} = \min\big(\arg\max_{x} \mu(x)\big), \tag{8}$$

$$\widehat{c} = \max\big(\arg\max_{x} \mu(x)\big), \tag{9}$$

respectively. If the membership function has only one global maxim, SOM and LOM will result in the same solution. If it has multiple global maxima, their solutions differ.

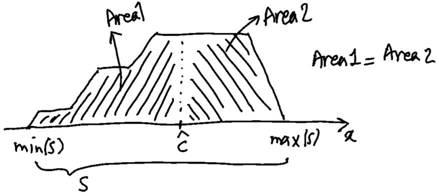


Defuzzification in fuzzy system

- The **bisector of area method** for defuzzification makes the following crisp value $\widehat{c}$ from the membership function $\mu(x)$ as follows:

$$\int_{\min(S)}^{\widehat{c}} \mu(x)dx = \int_{\widehat{c}}^{\max(S)} \mu(x)dx, \tag{10}$$

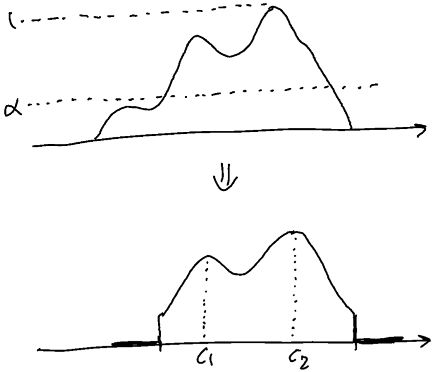
where S is the support set and $\widehat{c}$ is calculated by solving this equation for $\widehat{c}$.



Defuzzification in fuzzy system

- The **threshold method** for defuzzification is used in combination of any other defuzzification method. It uses the α-cut of the membership function, introduced before, rather than the original membership function, in other methods for defuzzification.

Example: threshold method along with the mean of maxima method



Fuzzy reasoning

Crisp relation versus fuzzy relation

- Consider a crisp function, or a crisp relation, as follows:

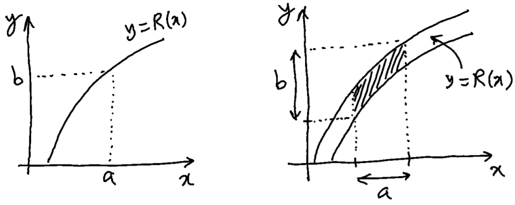
$$y = R(x). \tag{11}$$

For this crisp relation, x is a value and y will be a corresponding value. Therefore, the crisp relation is a curve.

- Now, consider a fuzzy relation as follows:

$$y = R(x). \tag{12}$$

For this fuzzy relation, x is a range of values and y will be a corresponding range of values. Therefore, the fuzzy relation is an interval-values function.



Fuzzy reasoning & composition rule of inference

- **Fuzzy reasoning**, also called approximate reasoning, is an inference procedure that derives conclusions from a set of if-then rules.
- The **composition rule of inference** is defined as follows.
- Recall the implication and the rule of inference introduced before. Let R be a fuzzy relation on $X \times Y$, i.e., $R(x, y) : X \to Y$, with the membership function $\mu_R(x, y)$.
- Consider the relation $R(x, y) : X \to Y$. Given the fuzzy set A in X to this relation will result in the fuzzy set B according to this relation.
- To obtain the resulting fuzzy set B, a cylindrical extension of A, i.e., $C(A)$ should be constructed. Let $\mu_{C(A)}(x)$ denote the membership function of $C(A)$.
- Then, the **composition operator** is defined and calculated as the T-norm of the the cylindrical extension of A and the relation, i.e., $\mu_{C(A)}(x) \wedge \mu_R(x, y) = \min\big(\mu_{C(A)}(x), \mu_R(x, y)\big)$. Note that the cylindrical extension is just for matching dimensions between A and R, so it is not an important thing.
- The antecedent fuzzy set B is found as the projection of the composition operator onto the fuzzy set Y which is the resulting set of relation:

$$\mu_B(y) = \bigvee_x \left(\mu_{C(A)}(x) \wedge \mu_R(x, y)\right) = \max_x \Big(\min\big(\mu_{C(A)}(x), \mu_R(x, y)\big)\Big). \tag{13}$$

- This composition is referred to as **max-min composition**.

Fuzzy reasoning & composition rule of inference

- Recall that the implication $A \rightarrow B$ in the fuzzy logic means as follows. The rule says if x is A, then y is B. Now, x is A'; therefore, y is B', where A' and B' can be far from or close to A and B, respectively.
- In the fuzzy logic, this inference is as follows:

$$B' = A' \circ R = A' \circ (A \rightarrow B), \tag{14}$$

where $\circ$ denotes giving A' as the input to the fuzzy relation R.
- According to the max-min composition, if x is A', then the antecedent B' is calculated as:

$$\mu_{B'}(y) = \bigvee_{x} \big(\mu_{A'}(x) \wedge \mu_R(x, y)\big) = \max_{x}\Big(\min\big(\mu_{A'}(x), \mu_R(x, y)\big)\Big). \tag{15}$$

Fuzzy reasoning: single rule with single input

- Consider a single rule with a single input. In this case, we have:
  - rule: if x is A, then y is B
  - fact: x is A'
  - conclusion: y is B'
- The relation is $R : A \to B$ so its membership function is:

$$\mu_R(x, y) = \mu_A(x) \wedge \mu_B(y). \tag{16}$$

  The inference is:

$$B' = A' \circ R = A' \circ (A \to B). \tag{17}$$

- In this case, the membership grade of inference is:

$$\begin{aligned}\mu_{B'}(y) &= \bigvee_x \left(\mu_{A'}(x) \wedge \mu_R(x, y)\right) \overset{(16)}{=} \bigvee_x \Big(\mu_{A'}(x) \wedge \left(\mu_A(x) \wedge \mu_B(y)\right)\Big) \\ &= \underbrace{\bigvee_x \left(\mu_{A'}(x) \wedge \mu_A(x)\right)}_{w} \wedge \mu_B(y) = w \wedge \mu_B(y),\end{aligned} \tag{18}$$

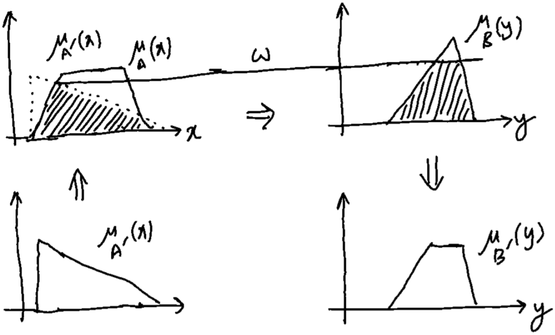
  where w is the **degree of validity** defined as:

$$w := \bigvee_x \left(\mu_{A'}(x) \wedge \mu_A(x)\right). \tag{19}$$

Fuzzy reasoning: single rule with single input

We found:

$$\mu_{B'}(y) = \underbrace{\bigvee_x (\mu_{A'}(x) \wedge \mu_A(x))}_{w} \wedge \mu_B(y) = w \wedge \mu_B(y).$$



Fuzzy reasoning: single rule with multiple inputs

- Consider a single rule with a multiple inputs. In this case, we have:
  - rule: if x is A and y is B, then z is C
  - fact: x is A' and y is B'
  - conclusion: z is C'
- The relation is $R : A \times B \rightarrow C$ so its membership function is:

$$\mu_R(x, y, z) = \mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z). \tag{20}$$

The inference is:

$$C' = (A' \times B') \circ R = (A' \times B') \circ (A \times B \rightarrow C). \tag{21}$$

Fuzzy reasoning: single rule with multiple inputs

- In this case, the membership grade of inference is:

$$\begin{aligned}
\mu_{C'}(z) &= \bigvee_{x,y} \Big((\mu_{A'}(x) \wedge \mu_{B'}(y)) \wedge \mu_R(x,y,z) \Big) \\
&= \bigvee_{x,y} \Big((\mu_{A'}(x) \wedge \mu_{B'}(y)) \wedge (\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)) \Big) \\
&= \bigvee_{x,y} (\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)) \wedge \mu_C(z) \\
&= \underbrace{\bigvee_{x} (\mu_{A'}(x) \wedge \mu_A(x))}_{w_1} \wedge \underbrace{\bigvee_{y} (\mu_{B'}(y) \wedge \mu_B(y))}_{w2} \wedge \mu_C(z) \\
&= \underbrace{(w_1 \wedge w_2)}_{w} \wedge \mu_C(z). \qquad (22)
\end{aligned}$$

where w_1 and w_2 are the degrees of validity of x and y, respectively, and w is the firing strength:

$$w_1 := \bigvee_{x} (\mu_{A'}(x) \wedge \mu_A(x)), \qquad (23)$$

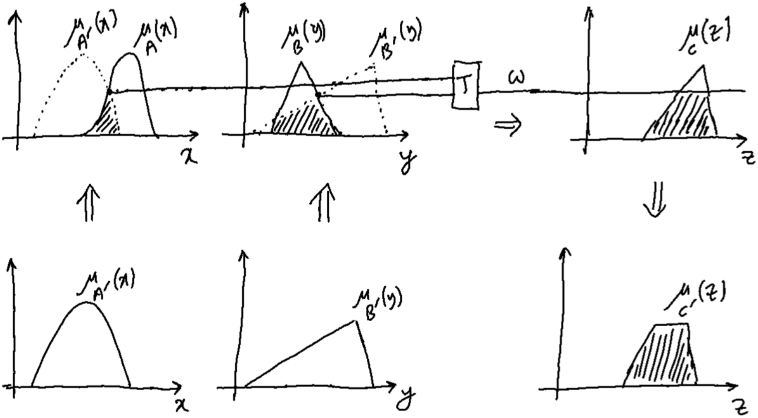
$$w_2 := \bigvee_{x} (\mu_{B'}(y) \wedge \mu_B(y)), \qquad (24)$$

$$w := w_1 \wedge w_2. \qquad (25)$$

Fuzzy reasoning: single rule with multiple inputs

- In general, if there are n inputs, the membership of the consequence of inference is:

$$\mu_{C'}(y) = \underbrace{(w_1 \wedge w_2 \wedge \cdots \wedge w_n)}_{w} \wedge \mu_C(z). \tag{26}$$



Fuzzy reasoning: multiple rules with multiple inputs

- Consider multiple rules with multiple inputs. In this case, we have:
  - rule 1: if x is A_1 and y is B_1, then z is C_1
  - rule 2: if x is A_2 and y is B_2, then z is C_2
  - fact: x is A' and y is B'
  - conclusion: z is C'
- The relations are $R_1 : A_1 \times B_1 \to C_1$ and $R_2 : A_2 \times B_2 \to C_2$, so their membership functions are:

$$\mu_{R_1}(x, y, z) = \mu_{A_1}(x) \wedge \mu_{B_1}(y) \wedge \mu_{C_1}(z), \tag{27}$$

$$\mu_{R_2}(x, y, z) = \mu_{A_2}(x) \wedge \mu_{B_2}(y) \wedge \mu_{C_2}(z). \tag{28}$$

- The total rule is the union (S-norm or maximum) of all rules:

$$R = R_1 \vee R_2. \tag{29}$$

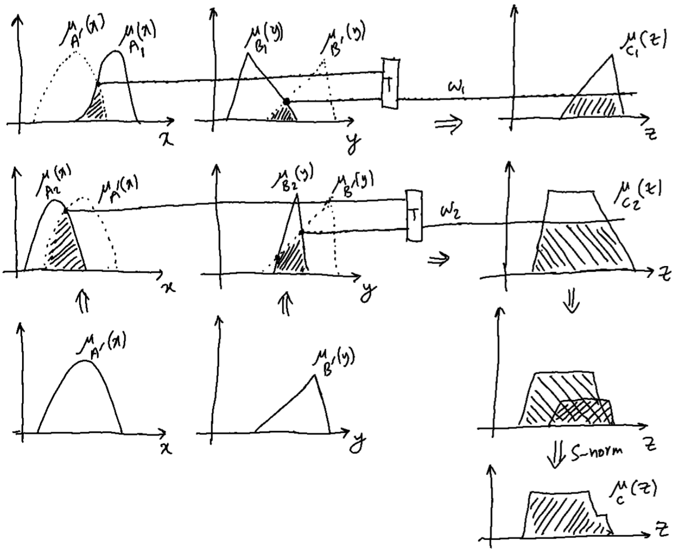
  The inference is:

$$C' = (A' \times B') \circ (R_1 \vee R_2) = \big((A' \times B') \circ R_1\big) \vee \big((A' \times B') \circ R_2\big) = C_1' \vee C_2', \tag{30}$$

  where $\big((A' \times B') \circ R_1\big)$ and $\big((A' \times B') \circ R_2\big)$ are each a rule with multiple inputs, introduced before. C_1' and C_2' are each the consequent of the two rules. The total consequence is the union (S-norm or maximum) of the consequences of the rules.

Fuzzy reasoning: multiple rules with multiple inputs

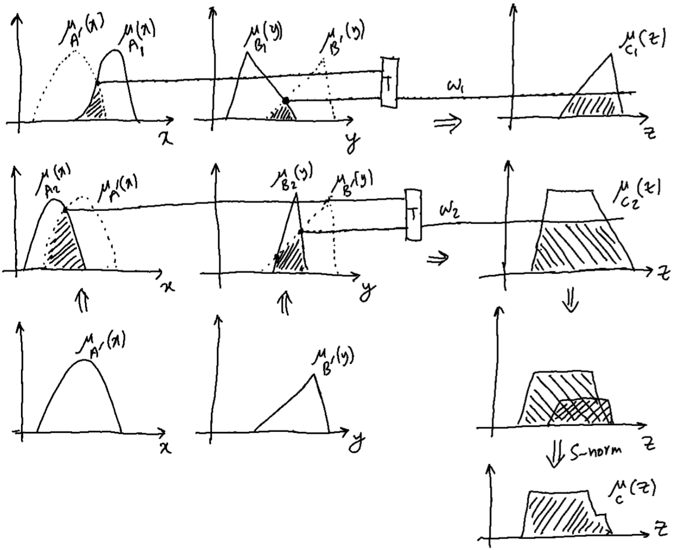
- In general, for r rules, we have: $C' = C'_1 \vee C'_2 \vee \cdots \vee C'_r$.



Mamdani fuzzy system

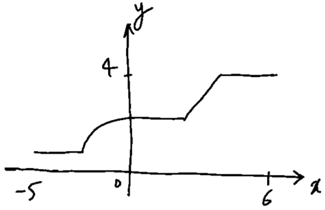
Mamdani fuzzy system

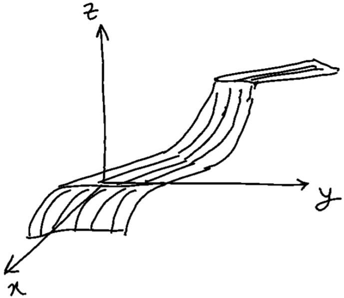
- In **Mamdani** fuzzy model, proposed in 1974 [1], the **max-min composition** is used.



Input-output curve

- It is possible to break down the input range with fine steps. Then, we calculate the defuzzified output y for every input x.
- The output y is obtained from the fuzzy inference system, such as the Mamdani model.
- This provides an input-output curve where the the corresponding output of the fuzzy system is known for every input of the fuzzy system.
- Therefore, fuzzy system is run only once for each input value and we do not to run it again every time. It makes the process faster in practice.





Sugeno fuzzy system

Sugeno fuzzy system

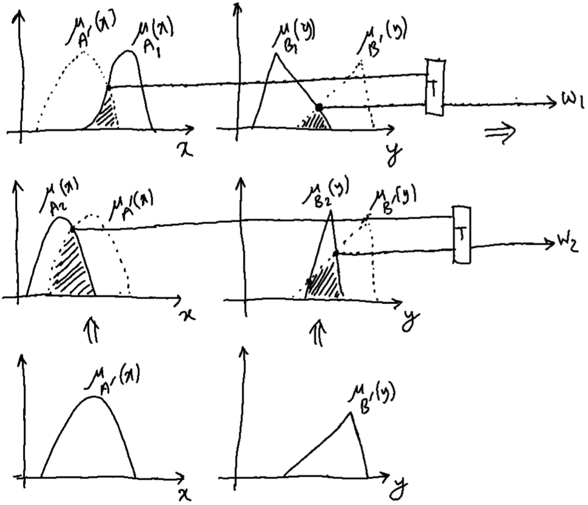
- The Sugeno fuzzy system, also called the Takagi-Sugeno-Kang (TSK) fuzzy system, was proposed in 1985 [2].
- The consequent in the Sugeno fuzzy model is a function of the antecedent. In other words, the defuzzification process is included in the execution of the fuzzy rules.
- For example, the fuzzy rules in this system are in this form:
  - If x is A_1 and y is B_1, then $z = f_1(x, y)$,
  - If x is A_2 and y is B_2, then $z = f_2(x, y)$,

  where every $f_i(.,.), \forall i$ is a crisp function.
- Usually, the polynomial functions are used in the Sugeno fuzzy system.
  - Sugeno zero-order model: $f_i(x, y) = c_i$,
  - Sugeno first-order model: $f_i(x, y) = a_i x + b_i y + c_i$,
  - and so on.
- An example for one input x and one output y is as follows:
  - If x is small, then $y = 0.1x + 3.2$
  - If x is medium, then $y = 0.5x + 2$
  - If x is large, then $y = x - 1$
- The overall output (aggregation) z is obtained through a weighted average of the outputs of each rule:

$$z = \frac{\sum_{i=1}^{n} w_i z_i}{\sum_{i=1}^{n} w_i}, \tag{31}$$

  where n is the number of rules, z_i is the output of the i-th rule, and w_i is the firing strength for the i-th rule.
- As a result, Sugeno fuzzy system does not need any defuzzifier as its output is already crisp.

Sugeno fuzzy system



$$z_1 = 0.1x + 6.5$$

$$z_2 = 2x^2 + 3x - 1$$

$$\Downarrow$$

$$z = \frac{\omega_1 z_1 + \omega_2 z_2}{\omega_1 + \omega_2}$$

Tsukamoto fuzzy system

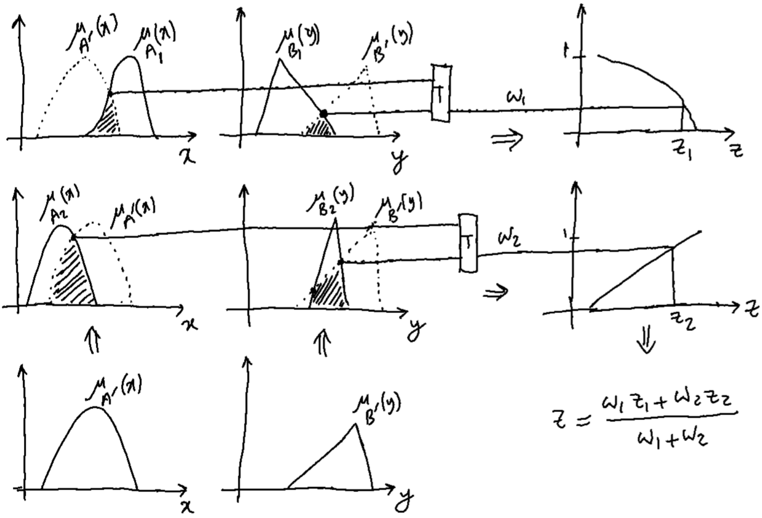
Tsukamoto fuzzy system

- In the Tsukamoto fuzzy system, the consequent of every fuzzy rule is obtained as the value in which the membership function of consequence equals the firing strength. In other words, the consequent of every fuzzy rule is the inverse of its membership function when it is equal to the firing strength of that rule.
- Of course, the output membership function should be a monotonic mapping, either increasing or decreasing, in this fuzzy system.
- The overall output (aggregation) is obtained through a weighted average:

$$y = \frac{\sum_{i=1}^{n} w_i z_i}{\sum_{i=1}^{n} w_i}, \quad (32)$$

  where n is the number of rules, z_i is the output of the i-th rule, and w_i is the firing strength for the i-th rule.
- As a result, Tsukamoto fuzzy system does not need any defuzzifier as its output is already crisp.

Tsukamoto fuzzy system



Advanced Topics in fuzzy logic

Advanced Topics in fuzzy logic

- Type-2 fuzzy sets and systems (1975 by Zadeh) [3]
  - In type-2 fuzzy sets, the membership function has also uncertainty.
- Complex fuzzy logic (2003) [4, 5, 6]
- Quaternion fuzzy logic (2020) [7]

Fuzzy systems in programming languages

Fuzzy systems in programming languages

- Fuzzy system in MATLAB:
  - Fuzzy Logic Designer
  - Fuzzy Logic in Simulink
  - `https://www.mathworks.com/products/fuzzy-logic.html`
- Fuzzy system in Python: scikit-fuzzy (or skfuzzy) library:
  - `https://pythonhosted.org/scikit-fuzzy/overview.html`
  - `https://pypi.org/project/scikit-fuzzy/`
  - `https://github.com/scikit-fuzzy/scikit-fuzzy`
  - `https://pythonhosted.org/scikit-fuzzy/api/skfuzzy.html`

Acknowledgment

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  - Prof. Mohammad Bagher Menhaj at the Amirkabir University of Technlogy, Department of Electrical Engineering
  - Prof. Saeed Bagheri Shouraki at the Sharif University of Technlogy, Department of Electrical Engineering

- Some good books in the area of fuzzy logic are:
  - George Klir, Bo Yuan, "Fuzzy Sets and Fuzzy Logic: Theory and Applications", 1995 [8]
  - Lotfi A. Zadeh, George J Klir, Bo Yuan, "Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers", 1996 [9]
  - Fakhreddine O Karray, Clarence W De Silva, "Soft computing and intelligent systems design: theory, tools, and applications", 2004 [10]
  - Merrie Bergmann, "An introduction to many-valued and fuzzy logic: semantics, algebras, and derivation systems", 2008 [11]
  - Vilém Novák, Irina Perfilieva, Jiri Mockor, "Mathematical principles of fuzzy logic", 2012 [12]

Acknowledgment

- Some important scholars in the area of fuzzy logic:
  - Lotfi A. Zadeh: https://scholar.google.com/citations?user=S6H-0RAAAAAJ&hl=en&oi=ao
  - Ebrahim Mamdani: https://en.wikipedia.org/wiki/Ebrahim_Mamdani
  - Michio Sugeno: https://scholar.google.com/citations?hl=en&user=RHzotX4AAAAJ
  - Tsukamoto
  - Ronald R Yager: https://scholar.google.com/citations?user=uAsllJMAAAAJ&hl=en
  - Jan Lukasiewicz: https://en.wikipedia.org/wiki/Jan_%C5%81ukasiewicz
  - Scott Dick (in the area of complex fuzzy, at the University of Alberta): https://scholar.google.com/citations?hl=en&user=9nMixQwAAAAJ
    - ★ See his interesting papers: [5, 6, 13, 14]
- Combination of Fuzzy logic and neural networks:
  - Adaptive Neuro Fuzzy Inference System (ANFIS) (1991-1993) [15, 16]
  - Complex ANFIS (ANCFIS) (2010) [14]
- My paper for use of fuzzy logic in stock prediction (fuzzy investment counselor): [17]
- Important journals in the area of fuzzy logic:
  - IEEE Transactions on Fuzzy Systems
  - Fuzzy Sets and Systems, Elsevier
  - Knowledge-Based Systems, Elsevier
  - Applied Soft Computing, Elsevier
  - Soft Computing, Springer
  - Fuzzy Optimization and Decision Making, Springer

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[1] E. H. Mamdani, "Application of fuzzy algorithms for control of simple dynamic plant," in *Proceedings of the institution of electrical engineers*, vol. 121, pp. 1585–1588, IET, 1974.

[2] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE transactions on systems, man, and cybernetics*, no. 1, pp. 116–132, 1985.

[3] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning–I," *Information sciences*, vol. 8, no. 3, pp. 199–249, 1975.

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