Game Theory:
Introduction & Game Elements

Adaptive and Cooperative Algorithms (ECE 457A)

ECE, MME, and MSCI Departments,
University of Waterloo, ON, Canada

Course Instructor: Benyamin Ghojogh
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Elements of a Game

Elements of a Game

- **Game Theory**: actions of decision makers who are conscious that their actions effect each other.
- **Players (agents)**: the individual who make decisions.
- **Action**: a choice made by a player.
- **Payoff (Utility)**: the utility or benefit received by a player either at the end of the game or in expectation before playing the game.
- **Information**: the amount of information that every player has about the game.
- **Strategy**: the plan a player uses to maximize its payoff. It is a subset of possible actions taken depending on the information available.
  - The strategy of the i-th player is denoted by s_i.
  - The payoff of the i-th player is denoted by $\pi_i(s_1, \ldots, s_n)$ where n is the number of players. This means that based on the chosen strategies of the players, the payoff of every player is determined.

Elements of a Game

- **Equilibrium**: The set of all selected strategies selected by each player is known as the equilibrium.
- **Outcome**: the equilibrium strategies dictate the outcome of the game.
- **Nature**:
  - a pseudo- and non-strategic player whose actions are taken purely mechanically.
  - taking random actions at specific points in the game with known probabilities.
  - each action by nature leads to different realizations or possibilities of the game
- **Action set**: the set of all possible actions for a player, denoted by $\mathcal{A}_i := \{a_i\}$ for the i-th player
- **Action profile**: a list of all actions taken by all players, denoted by $\mathcal{A} := \{a_i \mid i \in \{1, \ldots, n\}\}$ ~~for the i-th player~~.

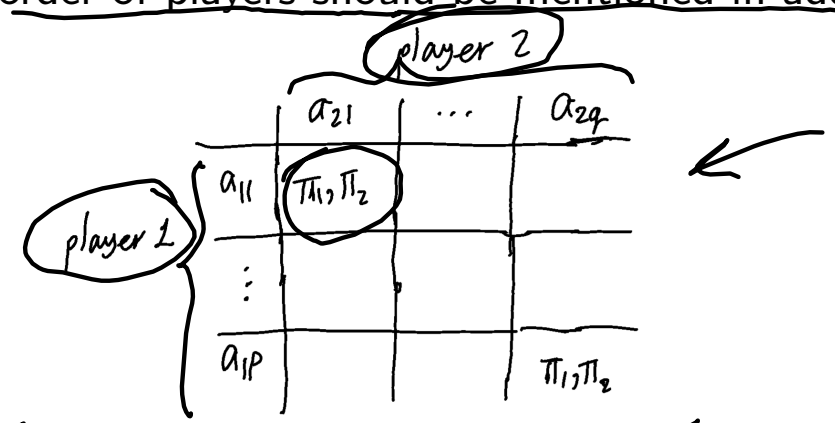
Cooperative versus Noncooperative Games

- **Cooperative game**: a game in which the players can cooperate and make binding commitments.
- **Noncooperative game**: a game in which the players cannot make binding commitments.
- In the noncooperative games, the players are **selfish** and want to maximize their own payoff only.
- Four cases may exist [1]:
  - **Cooperative game without conflict**: they collaborate to gain mutually.
  - **Cooperative game with conflict**: such as bargaining over price.
  - **Noncooperative game with conflict**: such as the prisoner's dilemma or the game of chicken (we will see them).
  - **Noncooperative game without conflict**: for example, two companies set a product standard without communication.

Representation Forms of a Game

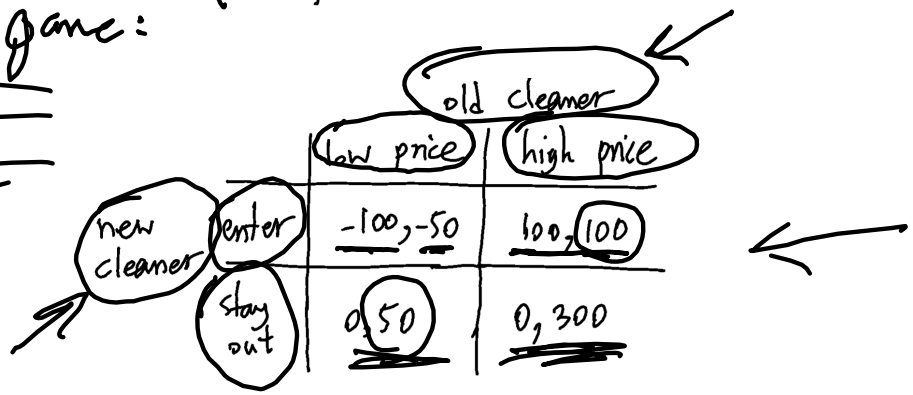
Normal Form

- There are two important forms of representing a game:
  - Normal form
  - Extensive form
- In the **normal form**, the game is shown as a **table** or **payoff matrix** where the rows and columns correspond to the actions of the first and second players, respectively.
- The normal form can be a multi-dimensional **tensor** if there are more than two (such as three) players.
- The normal form can be either for the **simultaneous-move game** or **ordered (sequential-move) games**.
- For ordered games, the order of players should be mentioned in addition to the table.



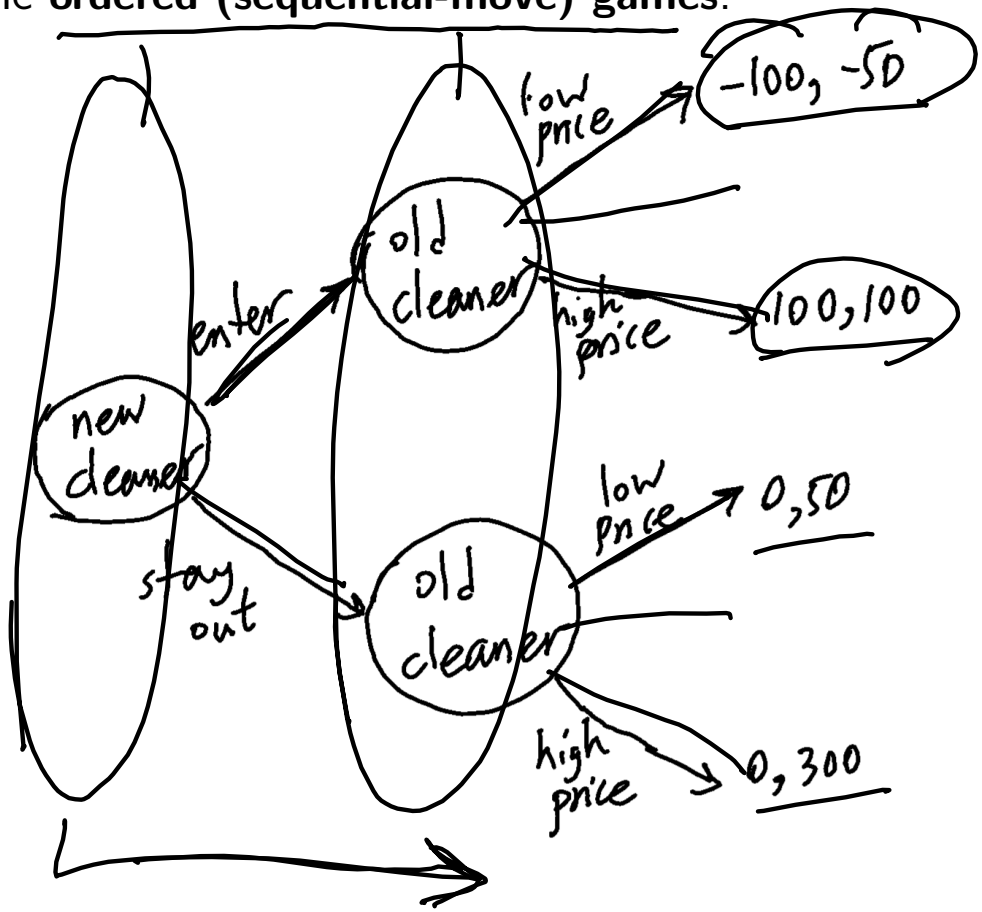
Example:

order of game:



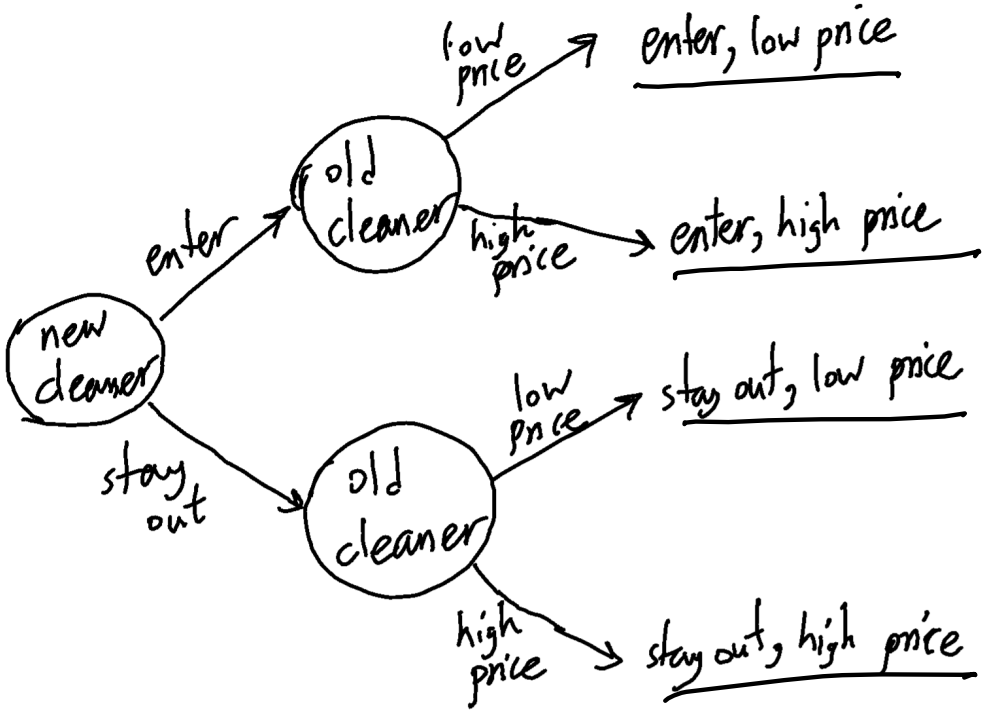
Extensive Form

- In the **extensive form**, the game is shown as a **tree**:
  1. A configuration of **nodes** and **branches** running without any closed loops from a **single starting node** to its **end nodes**.
  2. An indication of **which node** belongs to **which player**.
  3. The **probabilities** that **nature** uses to choose different branches at its nodes.
  4. The **payoffs** for each player at each end node.
- It is more suitable for the **ordered (sequential-move) games**.



Game Tree

- The **game tree** is similar to the extensive form but it has the game's **outcomes** rather than payoffs for each player at each end node.



Some Well-known Two-Person Games

Some Well-known Two-Person Games

- There are various games some of which are more well-known.
- A game may be explained in an example situation but it can be used to describe a backbone scenario in various situations.
- In other words, we observe and deal with these games in our daily lives.
- Some of the well-known two-person games are introduced in the following.

The prisoner's dilemma

- The **prisoner's dilemma** (1993) [2]:
  - two criminals are under arrest.
  - policemen talk to two prisoners separately.
  - if a prisoner cooperates (confesses to be guilty) and the other prisoner defects (denies being guilty), the cooperating prisoner will be enjailed for 3 years and the other one is freed.
  - if both cooperate (confess being guilty), they each are enjailed for 1 year.
  - if both defect (deny being guilty), they each are enjailed for 2 years.
  - The number of years may change in the game but their relative relation should be preserved.

		prisoner 2: cooperate	prisoner 2: defect
prisoner 1	cooperate	-1, -1	-3, 0
	defect	0, -3	-2, -2

The game of chicken

- The **game of chicken**, also known as the **hawk-dove game** or **snowdrift game** [3]:
  - two teenagers play a game where they drive their cars with a fast pace toward each other.
  - if one of them swerves and the other one continues driving, the one who continued wins the game.
  - if both continue driving until the end, they will crash and both die because of the accident.
  - if they both swerve, the both lose but will not die.

		teenager 2: continue	teenager 2: swerve
teenager 1	continue	−3, −3	2, 0
	swerve	0, 2	1, 1

- Similar scenario:
  - Grab the dollar game

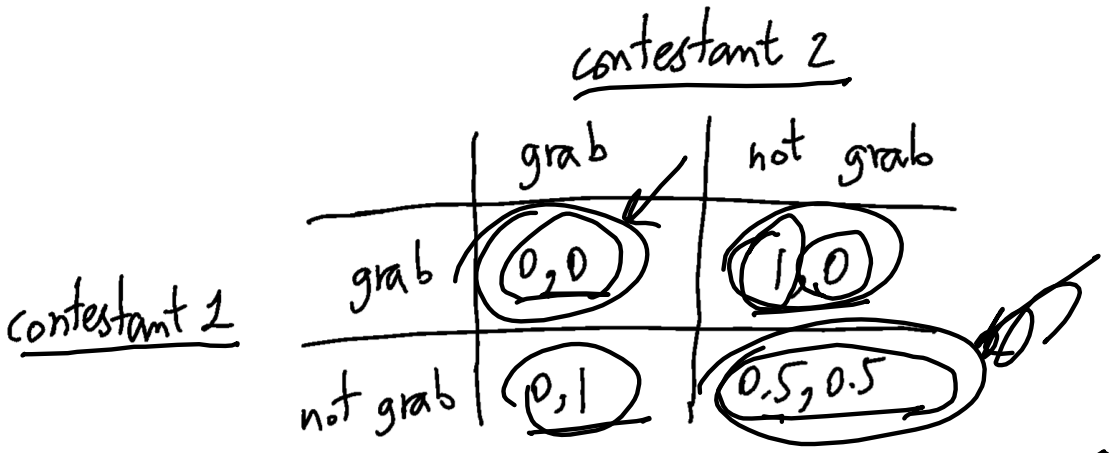
Grab the dollar game

- **Grab the dollar** game [1]:
  - there is a entertainment TV show, named "Let's Make a Deal", where two contestants should decide about some dollar as the reward.
  - each of them can either grab the dollar or not grab it.
  - if both grab the dollar, the lose the dollar and they are fined 1 dollar each.
  - if one of them grabs it and the other does not grab it, the person who grabbed it will earn the dollar.
  - if both of them do not grab the dollar, the lose the dollar.

		contestant 2	
		grab	not grab
contestant 1	grab	$-1, -1$	$1, 0$
	not grab	$0, 1$	$0, 0$

Grab the dollar game: another version

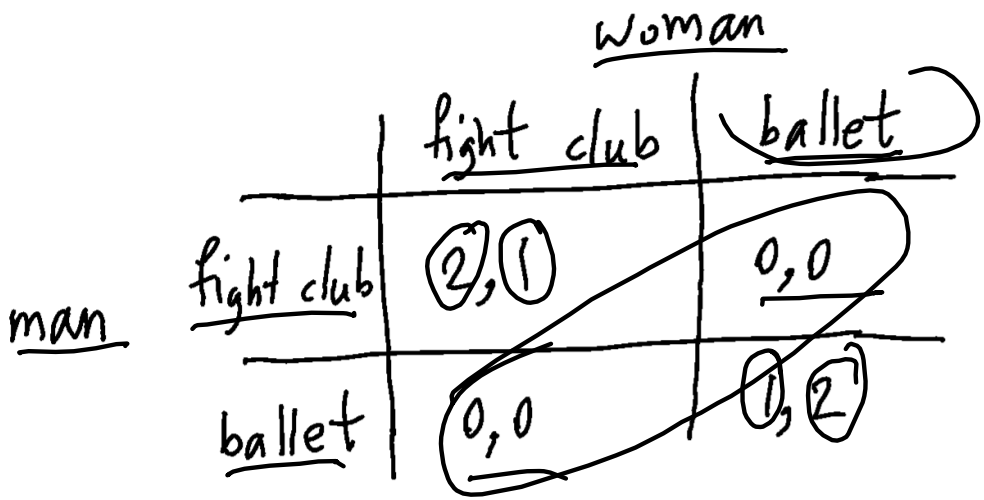
- Grab the dollar game (another version):
  - In the TV show "Let's Make a Deal", the game is the same as explained but with a slight difference.
  - if both of them do not grab the dollar, they split the dollar between each other.
  - if both grab the dollar, the lose the dollar but they are not fined.



- We will see that the Nash equilibria of this game are (grab, not grab) or (not grab, grab) if there is a fine for (grab, grab). If there is no fine, the game has three Nash equilibria, i.e., (grab, not grab), (not grab, grab), and (grab, grab).
- In the TV show, many contestants chose the strategy (not grab, not grab) to split the reward; although that is not the Nash equilibrium.
- However, once in the game, someone said to the other contestant: "I will grab the money. You don't grab it; otherwise, we both lose the money." The other contestant was shocked and said let's split the money but that contestant never agreed. That was a funny episode of the game.

The battle of sexes

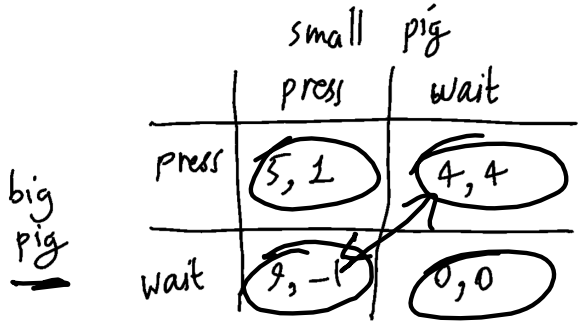
- The battle of sexes (1957) [4]:
  - a couple of man and woman decide to go out.
  - the man wants to go to the fight club show and the woman likes to go to a ballet show.
  - if both go to the fight club show, the man gets happy and if they both go to the ballet show, the woman gets happy.
  - if they cannot consensus, they do not go out at all.



The boxed pigs

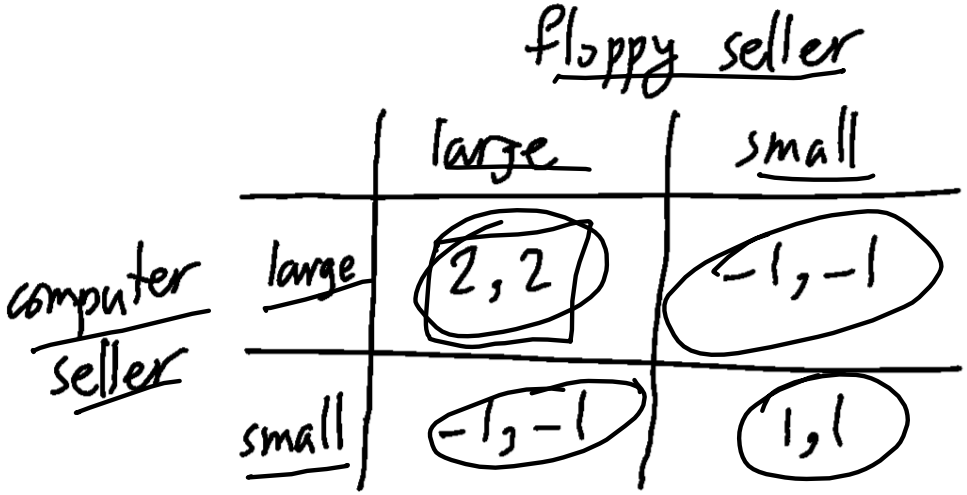
not symmetric game

- The boxed pigs [1]:
  - two pigs are put in a box with a control panel at one end and a food dispenser at the other end.
  - one pig is big and dominant and the other is small.
  - the actions are pressing the control panel or waiting for the other pig to press the control panel.
  - when a pig presses the panel, at a utility cost of 2 units, 10 units of food are dispensed at the dispenser.
  - (press, press): payoff of the big pig = 5 (10 units of food, minus 3 that the small pig eats, minus an effort cost of 2). The payoff of the small pig = 1 (3 units of food, minus an effort cost of 2).
  - (wait, press): the small pig presses so the big pig gets to the dispenser first and eats 9 units. The small pig eats its leavings, worth 1 unit, but pressing costs it 2 units (so total payoff is 1 - 2 = -1).
  - (press, wait): the big pig presses so the small pig gets to the dispenser first and eats 4 units. The big pig eats 6 units, but pressing costs it 2 units (so total payoff is 6 - 2 = 4).
  - (wait, wait): none of the pigs press the panel, so no food will be in the dispenser, so no one eats food.



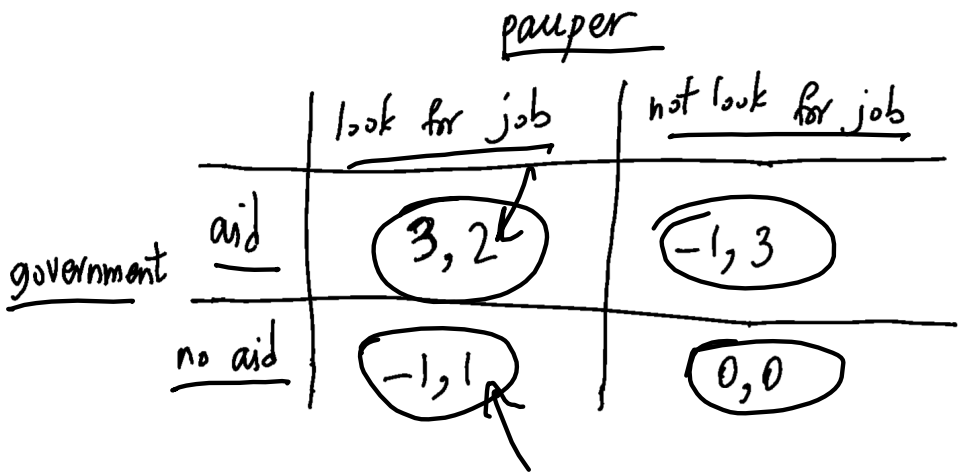
The ranked coordination

- The ranked coordination [5]:
  - two players are the managers of two computer companies.
  - One of the companies sells computers and the other company sells floppy disks.
  - player 1 decides whether to design computers accepting small or large floppy disks.
  - player 2 decides whether to design small or large floppy disks.
  - Large floppy disks is more profitable.



The welfare game

- The welfare game [1]:
  - a social-democrat government wants to aid a pauper if he/she is unemployed but looking for a job.
  - a pauper decides to look for a job or ignores looking for a job while getting the aid from the government.
  - If the government gives aid and the pauper look for a job, it is good for both.
  - If the government gives aid but the pauper does not look for a job, it is good for the pauper but not the government.
  - If the government does not give aid but the pauper looks for a job, it is neither good for the pauper nor for the government. Although, looking for a job is slightly good for the pauper.
  - If the government does not give aid and the pauper does not look for a job, it is a neutral case for both, i.e., they are both ignorant.



The battle of the Bismarck Sea

- The battle of the Bismarck Sea [1]:
  - the Battle of the Bismarck Sea, during the World War II, in the South Pacific in 1943.
  - General Imamura, in one side of the war, has been ordered to transport his troops across the Bismarck Sea to New Guinea. General Kenney, in another side of the war, wants to bomb the troop transports.
  - Imamura should choose between a shorter northern path or a longer southern path to New Guinea.
  - Kenney should decide where to send his planes to look for the Japanese.
  - If Kenney sends his planes to the wrong path, he can recall them but the number of days of bombing will be reduced.
  - Therefore, if both choose the same path, Kenny benefits and it is bad for Imamura. If they choose different paths, Imamura benefits and it is bad for Kenny (because Kenny can still recall his planes).
  - In all cases, the payoff of Imamura is negative because his troops will be bombed anyways but in some cases, they are less bombed.
  - Also, north is usually better for Imamura because the northern path is shorter.

		Imamura: north	Imamura: south
Kenney	north	2, −2	2, −2
	south	1, −1	3, −3

Zero-sum games

- **Zero-sum games**: one player's gain is with the other player's complete loss. As a result, the summation of payoffs of the players is zero.
- An example is the game of battle of the Bismarck Sea:

		Imamura: north	Imamura: south
Kenney	north	2, −2	2, −2
Kenney	south	1, −1	3, −3

Acknowledgment

- Some slides of this slide deck are inspired by teachings of Prof. Stanko Dimitrov at the University of Waterloo, Department of Management Science and Engineering.
- Some slides of this slide deck are based on the following book: Eric Rasmusen, "Games and Information: An Introduction to Game Theory", 4th Edition, 2007, [1]
  `https://www.rasmusen.org/GI/download.htm`
- A good lecture series on YouTube, by William Spaniel, about fundamentals of game theory (named "Game Theory 101: Strategic Form Games"):
  `https://www.youtube.com/playlist?list=PL7F0C4C7A4C910AF5`
- An important scholar in the area of game theory: Martin J. Osborne, who used to be a professor at the University of Toronto.
  Google Scholar:
  `https://scholar.google.com/citations?user=lx-4Hd8AAAAJ&hl=en&oi=sra`

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[4] R. D. Luce and H. Raiffa, *Games and decisions: Introduction and critical survey*.
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