Game Theory: Mixed Strategy

Adaptive and Cooperative Algorithms (ECE 457A)

ECE, MME, and MSCI Departments,
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Introduction

Introduction

- So far, the action space was discrete and finite. But what if the actions are continuous, such as price, volume, etc.
- The strategies for discrete actions are **pure strategies** because the players choose one of actions each:

$$s_i : w_i \to a_i, \tag{1}$$

where s_i and a_i are the pure strategy and pure action of the i-th player and w_i is the realization of game which it responds to by the strategy.

- The strategies for continuous actions are **mixed strategies** because we may have a mixture of actions and not pure actions:

$$s_i : w_i \to m(a_i), \tag{2}$$

where s_i and a_i are the mixed strategy and pure action of the i-th player and w_i is the realization of game which it responds to by the strategy.

- Here, $m(a_i)$ is a probability density function (PDF) on the action a_i; in other words, it is the **probability** that the i-th player plays action a_i:

$$m \geq 0, \quad \int_{\mathcal{A}_i} m(a_i) da_i = 1. \tag{3}$$

- A **completely mixed strategy** puts positive probability on every action; therefore, $m > 0$.

The welfare game

- Consider the welfare game:

pauper

		look for job	not look for job
government	aid	3, 2	−1, 3
	no aid	−1, 1	0, 0

- It does not have a pure Nash equilibrium or a pure dominant strategy.
- However, it has a mixed Nash equilibrium.
- We can consider probabilities for playing the actions:
  - θ: probability of action "aid"
  - $1-\theta$: probability of action "no aid"
  - γ: probability of action "look for job"
  - $1-\gamma$: probability of action "not look for job"

pauper

		look for job(γ)	not look for job($1-\gamma$)
government	aid(θ)	3, 2	−1, 3
	no aid($1-\theta$)	−1, 1	0, 0

Nash Equilibrium in Mixed Strategy

Nash Equilibrium in Mixed Strategy

- In the mixed strategy, we can find the Nash equilibrium with two approaches:
  - first-order condition
  - payoff-equating method

First-order Condition

pauper

government	look for job (γ)	not look for job ($1-\gamma$)
aid (θ)	3, 2	−1, 3
no aid ($1-\theta$)	−1, 1	0, 0

- Expectation (expected value) for discrete random variable X:

$$\boxed{\mathbb{E}[X] = \sum_x x\mathbb{P}(x),} \tag{4}$$

  where x is the value(s) that the random variable X can take and $\mathbb{P}(x)$ is the probability for the taking value x.

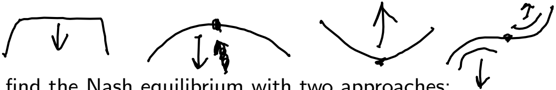
- The expected payoff for the government player:

$$\begin{aligned}\pi_{\text{government}} &= \theta\big(3\gamma + (-1)(1-\gamma)\big) + (1-\theta)\big((-1)\gamma + (0)(1-\gamma)\big)\\ &= \underbrace{5\theta\gamma - \theta - \gamma.}\end{aligned}$$

- The expected payoff for the pauper player:

$$\begin{aligned}\pi_{\text{pauper}} &= \gamma\big(2\theta + 1(1-\theta)\big) + (1-\gamma)\big(3\theta + (0)(1-\theta)\big)\\ &= \underbrace{-2\theta\gamma + \gamma + 3\theta.}\end{aligned}$$

First-order Condition



- In the mixed strategy, we can find the Nash equilibrium with two approaches:
  - first-order condition
  - payoff-equating method
- In the first-order condition method:
  - We use the fact that at the maximum of payoff, the gradient of payoff is zero.
  - We can also use second-order condition where the second-order derivative should be non-positive at the maximum.
- In the welfare game:

$$\pi_{\text{government}} = 5\theta\gamma - \theta - \gamma \implies \frac{\partial \pi_{\text{government}}}{\partial \theta} = 5\gamma - 1 \overset{\text{set}}{=} 0 \implies \boxed{\gamma = 0.2,}$$

$$\frac{\partial^2 \pi_{\text{government}}}{\partial \theta^2} = 0 \leq 0 \quad \checkmark$$

$$\pi_{\text{pauper}} = -2\theta\gamma + \gamma + 3\theta \implies \frac{\partial \pi_{\text{pauper}}}{\partial \gamma} = -2\theta + 1 \overset{\text{set}}{=} 0 \implies \boxed{\theta = 0.5,}$$

$$\frac{\partial^2 \pi_{\text{pauper}}}{\partial \gamma^2} = 0 \leq 0 \quad \checkmark$$

- So, in the equilibrium, government will aid with probability 0.5 (and will not aid with probability 0.5) and the pauper will look for a job with probability 0.2 (and will not look for a job with probability 0.8).

Payoff-Equating Method

pauper

government	look for job (γ)	not look for job ($1-\gamma$)
aid (θ)	3, 2	−1, 3
no aid ($1-\theta$)	−1, 1	0, 0

- In the payoff-equating method:
  - When a player uses a mixed strategy in equilibrium, it must get the same payoff from each of the pure strategies used in the mixed strategy.
  - Otherwise (if not equal), then the rational player plays the strategy with higher payoff more frequently, i.e., with probability 1.
- In the welfare game:

$$\pi_{\text{government}}(\text{aid}) = 3\gamma + (-1)(1-\gamma),$$
$$\pi_{\text{government}}(\text{no aid}) = (-1)\gamma + (0)(1-\gamma),$$
$$\pi_{\text{pauper}}(\text{look for job}) = 2\theta + 1(1-\theta),$$
$$\pi_{\text{pauper}}(\text{no look for job}) = 3\theta + 0(1-\theta).$$

Payoff-Equating Method

		pauper	
		look for job (γ)	not look for job ($1-\gamma$)
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- In the welfare game:

$$\pi_{\text{government}}(\text{aid}) = 3\gamma + (-1)(1-\gamma),$$
$$\pi_{\text{government}}(\text{no aid}) = (-1)\gamma + (0)(1-\gamma),$$
$$\pi_{\text{pauper}}(\text{look for job}) = 2\theta + 1(1-\theta),$$
$$\pi_{\text{pauper}}(\text{no look for job}) = 3\theta + 0(1-\theta).$$

- Pay-off equating method:

$$\boxed{\pi_{\text{government}}(\text{aid}) = \pi_{\text{government}}(\text{no aid})} \Longrightarrow 3\gamma + (-1)(1-\gamma) = (-1)\gamma + (0)(1-\gamma)$$
$$\Longrightarrow \underline{\gamma = 0.2},$$
$$\underline{\pi_{\text{pauper}}(\text{look for job}) = \pi_{\text{pauper}}(\text{no look for job})} \Longrightarrow 2\theta + 1(1-\theta) = 3\theta + 0(1-\theta)$$
$$\Longrightarrow \underline{\theta = 0.5}.$$

- So, in the equilibrium, government will aid with probability 0.5 (and will not aid with probability 0.5) and the pauper will look for a job with probability 0.2 (and will not look for a job with probability 0.8).

Categories of Games with Mixed Strategies

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- **Discoordination games**:

a, w	b, x
c, y	d, z

a, w	b, x
c, y	d, z

  - a single equilibrium in mixed strategies
  - the payoffs are:
    - ★ either $a > c$, $d > b$, $x > w$, $y > z$
    - ★ or $c > a$, $b > d$, $w > x$, $z > y$
  - example: the welfare game

pauper

government		look for job	not look for job
	aid	3, 2	−1, 3
	no aid	−1, 1	0, 0

Categories of Games with Mixed Strategies

- **Coordination games**:

a, w	b, x
c, y	d, z

  - three equilibria:
    - ⋆ two symmetric equilibria in pure strategies
    - ⋆ one symmetric equilibrium in mixed strategies
  - the payoffs are: $a > c$, $d > b$, $w > x$, $z > y$
  - example: the ranked coordination game

floppy seller

computer seller	large	small
large	2, 2	−1, −1
small	−1, −1	1, 1

Categories of Games with Mixed Strategies

- **Contribution games**:

	a_1	a_2
a_1	a, w	b, x
a_2	c, y	d, z

  - three equilibria:
    - ★ two asymmetric equilibria in pure strategies
    - ★ one symmetric equilibrium in mixed strategies
  - the payoffs are:
    - ★ $c > a$, $b > d$, $x > w$, $y > z$
    - ★ moreover, we have either $b > c$, $y > x$ or $c > b$, $x > y$

Acknowledgment

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- Some slides of this slide deck are based on the following book: Eric Rasmusen, "Games and Information: An Introduction to Game Theory", 4th Edition, 2007, [1] `https://www.rasmusen.org/GI/download.htm`

References

[1] E. Rasmusen, *Games and information: An introduction to game theory*. Wiley-Blackwell, 4 ed., 2007.