ECE 457A TUTORIAL 01: CONVEXITY

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Show that the following set is a convex set:

$$\{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}\},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{C} \in \mathbb{R}^{k\times n}$, $\boldsymbol{d} \in \mathbb{R}^k$.

Hint: Take two points in this set and use the definition of the convex set to show it is convex.

$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} \longrightarrow$ vectors

$Ax_1 \leq b \quad Cx_1 = d \quad \Longrightarrow \alpha \in [0,1]$

$Ax_2 \leq b \quad Cx_2 = d$

$A(\underbrace{\alpha x_1 + (1-\alpha)x_2}) = \alpha A x_1 + (1-\alpha) A x_2 \leq \underbrace{\alpha b + (1-\alpha) b}_{b}$

$\Longrightarrow A(\ \ ) \leq b$

$$C(\alpha x_1 + (1-\alpha)x_2) = \alpha C x_1 + (1-\alpha) C x_2$$
$$= \alpha d + (1-\alpha) d = d$$

So, $\alpha x_1 + (1-\alpha)x_2$ also exists in the set.

So, it is convex.

Show that the unit ball set, with norm $\|.\|$, is a convex set:

$$\mathcal{B}(\boldsymbol{x}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| \leq 1\}.$$

$f(x) \longrightarrow f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$ $\cancel{\alpha}$

Here $\longrightarrow f(x) = \|x\| \longrightarrow l_1$- norm

Previous $\longrightarrow \begin{cases} f_1(x) = Ax \\ f_2(x) = Cx \end{cases}$

$\left.\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} \longrightarrow$ vectors

$\alpha \longrightarrow$ scalar

$[0, 1]$

$\|x_1\| \leq 1$

$\|x_2\| \leq 1$

$\|\alpha x_1 + (1-\alpha) x_2\| \leq$

$\|\alpha x_1\| + \|(1-\alpha) x_2\| =$

$$\underbrace{\underbrace{\alpha \|x_1\|}_{\leq \alpha} + \underbrace{(1-\alpha)\|x_2\|}_{\leq 1-\alpha}}_{\leq \alpha + 1 - \alpha = 1}$$

Definition (Convex set and convex hull)

A set $\mathcal{D}$ is a convex set if it completely contains the line segment between any two points in the set $\mathcal{D}$:

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}, 0 \leq t \leq 1 \implies t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathcal{D}.$$

The convex hull of a (not necessarily convex) set $\mathcal{D}$ is the smallest convex set containing the set $\mathcal{D}$. If a set is convex, it is equal to its convex hull.

Definition (Convex function)

A function $f(.)$ with domain $\mathcal{D}$ is convex if:

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}, \tag{1}$$

where $\alpha \in [0, 1]$.

Definition (Convex function)

If the function $f(.)$ is differentiable, it is convex if:

$$f(\boldsymbol{x}) \geq f(\boldsymbol{y}) + \nabla f(\boldsymbol{y})^\top (\boldsymbol{x} - \boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}. \tag{2}$$

Definition (Convex function)

If the function $f(.)$ is twice differentiable, it is convex if its second-order derivative is positive semi-definite:

$$\nabla^2 f(\boldsymbol{x}) \succeq \boldsymbol{0}, \quad \forall \boldsymbol{x} \in \mathcal{D}. \tag{3}$$

Show that the following function is a convex function:

$$f(x) = x^2,$$

where $\boldsymbol{x} \in \mathbb{R}$.

Hint: Use any of the definitions of the convex function to show this function is convex.

$\boxed{S1}$ $x_1 \neq x_2$ $\alpha \in [0, 1]$

$f(\alpha x_1 + (1-\alpha) x_2) = (\alpha x_1 + (1-\alpha) x_2)^2 = \alpha^2 x_1^2 + 2\alpha(1-\alpha) x_1 x_2 + (1-\alpha)^2 x_2^2$

$x_1 \neq x_2 \Longrightarrow (x_1 - x_2)^2 > 0$

$x_1^2 + x_2^2 - 2 x_1 x_2 > 0 \Longrightarrow x_1^2 + x_2^2 > 2 x_1 x_2$

$$x_1^2 + x_2^2 > 2x_1x_2 \implies 2x_1x_2 < x_1^2 + x_2^2$$

$$f(\alpha x_1 + (1-\alpha)x_2) = \alpha^2 x_1^2 + \underbrace{2\alpha(1-\alpha)x_1x_2}_{< \alpha(1-\alpha)(x_1^2+x_2^2)} + (1-\alpha)^2 x_2^2$$

$$\implies \alpha^2 x_1^2 + (1-\alpha)^2 x_2^2 + 2\alpha(1-\alpha)x_1x_2 < \alpha^2 x_1^2 + (1-\alpha)^2 x_2^2 + \underbrace{\alpha(1-\alpha)(x_1^2+x_2^2)}_{*}$$

$$\implies * = \cancel{\alpha^2 x_1^2} + x_2^2 - 2\alpha x_2^2 + \cancel{\alpha^2 x_2^2} + \alpha x_1^2 \cancel{- \alpha^2 x_1^2} + \alpha x_2^2 \cancel{- \alpha^2 x_2^2}$$

$$= x_2^2 - 2\alpha x_2^2 + \alpha x_1^2 + \alpha x_2^2$$

$$= \alpha x_1^2 + (1-\alpha)x_2^2 = \alpha f(x_1) + (1-\alpha) f(x_2)$$

$\boxed{S2}$ $f(x_1) \geq f(x_2) + \nabla f(x_2)^T (x_1 - x_2)$

$\implies x_1^2 \geq x_2^2 + 2x_2(x_1 - x_2)$

$\implies x_1^2 \geq \underbrace{x_2^2 + 2x_1x_2 - 2x_2^2}_{-x_2^2 + 2x_1x_2}$

$\implies x_1^2 - 2x_1x_2 + x_2^2 \geq 0$

$\implies (x_1 - x_2)^2 \geq 0$ ✓

$\boxed{S3}$

$$f(x) = x^2 \Rightarrow \frac{\partial f}{\partial x} = 2x \Longrightarrow \frac{\partial^2 f}{\partial x^2} = 2 \geq 0 \quad \checkmark$$

$$\forall x \in \mathbb{R}$$

Show that the following function is a convex function:

$$f(\boldsymbol{x}) = \boldsymbol{a}^\top \boldsymbol{x} + b,$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{a} \in \mathbb{R}^n$, and $b \in \mathbb{R}$.

Hint: Use any of the definitions of the convex function to show this function is convex.

$a^T x = \text{scalar}$

$[a_1 \;\; a_2]\begin{bmatrix} x \\ y \end{bmatrix}$

$= a_1 x + a_2 y$

$= \text{scalar}$

$\boxed{S1}$ $\quad x_1 \neq x_2 \qquad \alpha \in [0, 1]$

$f(\alpha x_1 + (1-\alpha) x_2) = a^T [\alpha x_1 + (1-\alpha) x_2] + b$

$= \alpha a^T x_1 + (1-\alpha) a^T x_2 + b \underbrace{+\alpha b - \alpha b}_{0}$

$= \alpha [a^T x_1 + b] + (1-\alpha) [a^T x_2 + b]$

$= \alpha f(x_1) + (1-\alpha) f(x_2)$

So, the function is convex.

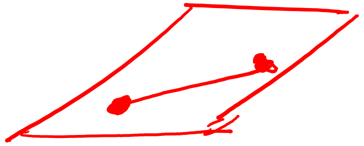
$$f(\alpha x_1 + (1-\alpha) x_2) = \alpha f(x_1) + (1-\alpha) f(x_2)$$

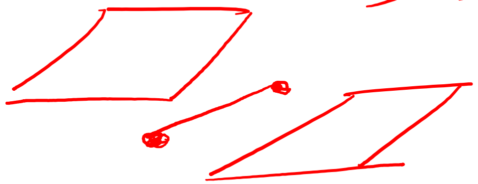
$\leq$

$\Rightarrow \leq$

$ax+b \longrightarrow$ line

$a^T x + b \longrightarrow$ plane





$\boxed{S2}$ $x_1 \neq x_2$

$$f(x_1) \geq f(x_2) + \nabla f(x_2)^T (x_1 - x_2)$$

$$\Rightarrow a^T x_1 + b \geq \underbrace{a^T x_2 + b + a^T (x_1 - x_2)}_{a^T x_1 + b}$$

$$\Rightarrow a^T x_1 + b = a^T x_1 + b \quad \checkmark$$

$\boxed{S3}$

$f(x) = a^T x + b$

$$\frac{\partial f}{\partial x} = a$$

$$\frac{\partial^2 f}{\partial x^2} = 0 = 0 \qquad \forall x \in \mathbb{R}^n \quad \checkmark$$

$\geq$