ECE 457A TUTORIAL 08: GAME THEORY

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Mixed Strategies (Example 1)

There is no NE in this game.

If we play this game, we should be unpredictable.

$\Longrightarrow$ We must randomize (mix) between strategies.

		P2	
		H (α)	T $(1-\alpha)$
P1	H (θ)	1, -1	-1, 1
	T $(1-\theta)$	-1, 1	1, -1

$\boxed{\text{First-Order Condition}}$

$$\pi_{P1} = (\theta)\Big(\underbrace{(1)(\alpha) + (-1)(1-\alpha)}_{2\alpha-1}\Big) + (1-\theta)\Big(\underbrace{(-1)(\alpha) + (1)(1-\alpha)}_{-2\alpha+1}\Big) = 4\alpha\theta - 2\theta - 2\alpha + 1$$

$$\pi_{P2} = (\alpha)\Big(\underbrace{(-1)(\theta) + (1)(1-\theta)}_{-2\theta+1}\Big) + (1-\alpha)\Big(\underbrace{(1)(\theta) + (-1)(1-\theta)}_{2\theta-1}\Big) = -4\alpha\theta + 2\theta + 2\alpha - 1$$

$$\frac{\partial \Pi_{P1}}{\partial \theta} = 4\alpha - 2 = 0 \implies \boxed{\alpha = 0.5}$$

$$\frac{\partial \Pi_{P2}}{\partial \alpha} = -4\theta + 2 = 0 \implies \boxed{\theta = 0.5}$$

$$\frac{\partial^2 \Pi_{P1}}{\partial \theta^2} = 0 \leq 0 \checkmark$$

$$\frac{\partial \Pi_{P2}}{\partial \alpha^2} = 0 \leq 0 \checkmark$$

Payoff-Equating Method

$$P1 : 2\alpha - 1 = -2\alpha + 1 \implies \boxed{\alpha = 0.5}$$

$$P2 : -2\theta + 1 = 2\theta - 1 \implies \boxed{\theta = 0.5}$$

Mixed Strategies (Example 2)

we need to get rid of as many strategies as we can.

For P1, M is completely dominated by U !!!

For P2, m is dominated by l !!!

	l	~~m~~	r
U	3, 2	~~2, 1~~	1, 3
~~M~~	~~2, 1~~	~~1, 5~~	~~0, 3~~
D	1, 3	~~4, 2~~	2, 2

	α	$(1-\alpha)$
θ	3, 2	1, 3
$(1-\theta)$	1, 3	2, 2

$\boxed{\text{FOC}}$

$$\pi_{p1} = \theta\Big(\underbrace{(3)(\alpha) + (1)(1-\alpha)}_{2\alpha+1}\Big) + (1-\theta)\Big(\underbrace{(1)(\alpha) + (2)(1-\alpha)}_{-\alpha+2}\Big) = 3\alpha\theta - \alpha - \theta + 2$$

$$\pi_{p2} = \alpha\Big(\underbrace{(2)(\theta) + (3)(1-\theta)}_{-\theta+3}\Big) + (1-\alpha)\Big(\underbrace{(3)(\theta) + (2)(1-\theta)}_{\theta+2}\Big) = -2\alpha\theta + \alpha + \theta + 2$$

$$\frac{\partial \pi_{p1}}{\partial \theta} = 3\alpha - 1 = 0 \Longrightarrow \boxed{\alpha = \frac{1}{3}}$$

$$\frac{\partial^2 \pi_{p1}}{\partial \theta^2} = 0 \leq 0 \checkmark$$

$$\frac{\partial \pi_{p2}}{\partial \alpha} = -2\theta + 1 = 0 \Longrightarrow \boxed{\theta = \frac{1}{2}}$$

$$\frac{\partial^2 \pi_{p2}}{\partial \alpha^2} = 0 \leq 0 \checkmark$$

$\boxed{\text{PEM}}$

$$\begin{cases} P1: (2\alpha+1) = (-\alpha+2) \Longrightarrow \boxed{\alpha = \frac{1}{3}} \checkmark \\ P2: (-\theta+3) = (\theta+2) \Longrightarrow \boxed{\theta = \frac{1}{2}} \checkmark \end{cases}$$

Continuous Strategies (Cournot and Stackelberg)

profit per quantity : $P(Q) = 9 - q_1 - q_2$

total output : $Q = q_1 + q_2$

marginal cost : $C(q) = 2q \longrightarrow \boxed{c = 2}$

payoffs : $\begin{cases} \pi_1 = P(Q)\,q_1 - C(q_1) = (9 - q_1 - q_2)\,q_1 - 2q_1 \\ \pi_2 = P(Q)\,q_2 - C(q_2) = (9 - q_1 - q_2)\,q_2 - 2q_2 \end{cases}$

Cournot

Reaction Functions

$$R_1 = \frac{\partial \pi_1}{\partial q_1} = 7 - 2q_1 - q_2 = 0$$

$$R_2 = \frac{\partial \pi_2}{\partial q_2} = 7 - q_1 - 2q_2 = 0$$

$$\xRightarrow[\text{Eq.}]{\text{Cournot-Nash}} R_1 = R_2 \Rightarrow \boxed{q_1 = q_2}$$

$$R_1 = 7 - 2q_1 - q_1 = 0 \Longrightarrow \boxed{q_1 = \frac{7}{3} = q_2} \checkmark$$

total market output $= Q = q_1 + q_2 = \frac{14}{3}$

Stackelberg

Let us assume that player 2 moves first. So, player 2 is the Stackelberg leader, while player 1 is the Stackelberg follower.

Therefore, player 2 predicts the reaction function of player 1.

$$R_1 = \frac{\partial \pi_1}{\partial q_1} = 7 - 2q_1 - q_2 = 0 \Rightarrow q_1 = \frac{7 - q_2}{2} \quad (*)$$

substitution

$$\pi_2 = \left(9 + \frac{q_2 - 7}{2} - q_2\right) q_2 - 2q_2$$

$$\Rightarrow \frac{\partial \pi_2}{\partial q_2} = 7 - \frac{7}{2} - q_2 = 0 \Rightarrow \boxed{q_2 = \frac{7}{2}} \xrightarrow{(*)} \boxed{q_1 = \frac{7}{4}}$$

$$Q = q_1 + q_2 = \frac{7}{4} + \frac{7}{2} = \frac{21}{4}$$

$$\begin{cases} P1 : \text{follower} \\ P2 : \text{leader} \end{cases} \Longrightarrow \begin{cases} q_1 = \frac{7}{4} \\ q_2 = \frac{7}{2} \end{cases} \Longrightarrow Q = \frac{21}{4}$$

$$\begin{cases} P1 : \text{leader} \\ P2 : \text{follower} \end{cases} \Longrightarrow \begin{cases} q_1 = \frac{7}{2} \\ q_2 = \frac{7}{4} \end{cases} \Longrightarrow Q = \frac{21}{4}$$

Cournot

$q_1 = q_2 = \frac{7}{3} = 2.37$

$Q = q_1 + q_2 = \frac{14}{3} = 4.67$

$P(Q) = 4.33$

Stackelberg

$q_1 = \frac{7}{4} = 1.75$

$q_2 = \frac{7}{2} = 3.5$

$Q = q_1 + q_2 = \frac{21}{4} = 5.25$

most total output

$P(Q) = 3.75$

lowest price

Consumers' preference

Continuous Strategies (Bertrand)

Two firms setting price at the same time

$$\begin{cases} q_1 = 72 - 3P_1 + 2P_2 \\ q_2 = 72 + 2P_1 - 3P_2 \end{cases} \qquad \begin{cases} \text{marginal cost} \\ c = 0 \quad \text{for both} \end{cases}$$

Payoffs

$$\begin{cases} \pi_1 = q_1 (P_1 - \overset{0}{\cancel{c}}) = 72P_1 - 3P_1^2 + 2P_1P_2 \\ \pi_2 = q_2 (P_2 - \overset{0}{\cancel{c}}) = 72P_2 + 2P_1P_2 - 3P_2^2 \end{cases}$$

$$\left.\begin{aligned} R_1 &= \frac{\partial \pi_1}{\partial P_1} = 72 - 6P_1 + 2P_2 = 0 \quad (*) \\ R_2 &= \frac{\partial \pi_2}{\partial P_2} = 72 + 2P_1 - 6P_2 = 0 \end{aligned}\right\} \Rightarrow R_1 = R_2$$

$$\Rightarrow -6P_1 + 2P_2 = 2P_1 - 6P_2 \Rightarrow \boxed{P_1 = P_2}$$

$$(*)\quad 72 - 6P_1 + 2P_1 = 0 \Rightarrow \boxed{P_1 = 18 = P_2}$$

$$q_1 = q_2 = 54 \Rightarrow \boxed{Q = q_1 + q_2 = 108}$$

References

- https://www.tayfunsonmez.net/wp-content/uploads/2013/10/E308SL7.pdf
- https://eml.berkeley.edu/~webfac/dellavigna/e101a_sp08/fexam3solutions.pdf

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