Integer Linear Programming

Optimization Techniques (ENGG*6140)

School of Engineering,
University of Guelph, ON, Canada

Course Instructor: Benyamin Ghojogh
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Integer Linear Programming

Integer linear programming

A **linear programming** problem is of the form:

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \text{linear function in } \boldsymbol{x} \\ \text{subject to} & \text{affine inequality constraints in } \boldsymbol{x}, \\ & \text{affine equality constraints in } \boldsymbol{x}. \end{cases}$$

An **integer linear programming** problem is of the form:

$$\begin{cases} \underset{\boldsymbol{x}}{\text{minimize}} & \text{linear function in } \boldsymbol{x} \\ \text{subject to} & \text{affine inequality constraints in } \boldsymbol{x}, \\ & \text{affine equality constraints in } \boldsymbol{x}. \\ & \boldsymbol{x} \in \mathbb{Z}. \end{cases}$$

In other words, integer linear programming is linear programming where the optimization variables are restricted to be **integer**.

If some of the variables are integer and some are not, we have mixed-integer programming [1].

Practical Example

- A company has two products. Let x_1 and x_2 denote the **number** of the first and second products to be produced, respectively. Therefore, $x_1, x_2 \geq 0$ and $x_1, x_2 \in \mathbb{Z}$.
- The company has profits \$60 and \$30 for the first and second products. Therefore, the total profit of company:

$$c = \$(60x_1 + 30x_2).$$

- The resources for these products are limited, so we have the following restrictions:
  - We do not want the first product, with proportion 8, and the second product, with proportion 3, to spend more than \$48, so: $8x_1 + 3x_2 \leq \$48$.
  - For four of the first product and three of the second product, we have the budget to spend at least \$25, so: $4x_1 + 2x_2 \geq \$25$.

The optimization becomes:

$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 60x_1 + 30x_2 \\ \text{subject to} \quad & 8x_1 + 3x_2 \leq 48, \\ & 4x_1 + 2x_2 \geq 25, \\ & x_1, x_2 \geq 0., \\ & x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

Integer linear programming

Problem 1

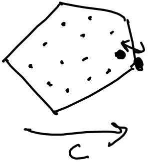
$$
\begin{aligned}
\underset{x_1,x_2}{\text{maximize}} \quad & \boxed{c = 5x_1 + 6x_2} \\
\text{subject to} \quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& x_1, x_2 \ge 0, \\
& x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

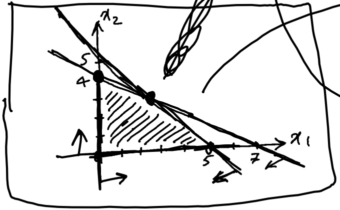
c_1^*

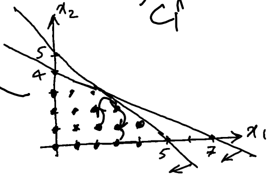
Problem 2

$$
\begin{aligned}
\max_{x_1,x_2} \quad & c = 5x_1 + 6x_2 \\
& x_1 + x_2 \le 5 \\
& 4x_1 + 7x_2 \le 28 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

c_2^*







$c_1^* \geq c_2^*$

$\boxed{c_1^* \leq c_2^*}$

$c_1^* = c_2^*$

Questions:

- Is **integer optimization** (such as integer linear programming) harder or easier than **continuous optimization** (such as linear programming)?
- Is the **optimum value of objective function** in integer linear programming better or worse than the that value in linear programming?

Subproblem 1

We relax the problem to linear programming:

$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 5, \\
& 4x_1 + 7x_2 \leq 28, \\
& x_1, x_2 \geq 0. \\
& \cancel{x_1, x_2 \in \mathbb{Z}}
\end{aligned}
$$

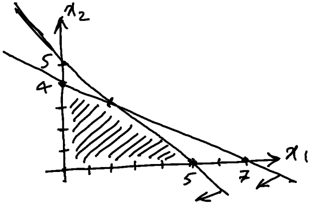
$$
\begin{aligned}
\max \quad & c = 5x_1 + 6x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 5 \\
& 4x_1 + 7x_2 \leq 28 \\
& x_1, x_2 \geq 0 \\
& x_1 \in \mathbb{Z}
\end{aligned}
$$

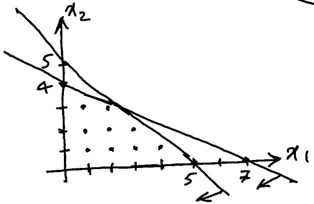
x_1 : integer

x_2 : float

Solution:

$$\boxed{c^* = 27.67, x_1^* = 2.33, x_2^* = 2.67.}$$





Subproblem 1

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 5, \\ & 4x_1 + 7x_2 \le 28, \\ & x_1, x_2 \ge 0. \end{aligned}$$

	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	5
s_2	4	7	0	1	28
c	-5	-6	0	0	0

min test:
$5/1 = 5$
$28/7 = 4$

		x_1	x_2	s_1	s_2	RHS
$r_1 - \frac{r_2}{7}$	s_1	$\frac{3}{7}$	0	1	$-\frac{1}{7}$	1
$r_2/7$	x_2	$\frac{4}{7}$	1	0	$\frac{1}{7}$	4
$r_4 + \frac{6}{7} r_2$	c	$\frac{-11}{7}$	0	0	$\frac{6}{7}$	24

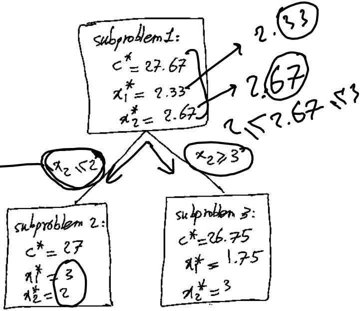
min test:
$\frac{1}{3/7} = \frac{7}{3}$
$\frac{4}{4/7} = 7$

		x_1	x_2	s_1	s_2	RHS	
$\frac{7}{3} r_1$	x_1	1	0	$\frac{7}{3}$	$\frac{-1}{3}$	$\frac{7}{3}$	$\Rightarrow x_1^* = \frac{7}{3} = 2.33 \leftarrow$
$r_2 - \frac{4}{3} r_1$	x_2	0	1	$\frac{4}{3}$	$\frac{7}{21}$	$\frac{8}{3}$	$\Rightarrow x_2^* = \frac{8}{3} = 2.67 \leftarrow$
$r_4 + \frac{11}{3} r_1$	c	0	0	$\frac{11}{3}$	$\frac{1}{3}$	$\frac{83}{3}$	$\Rightarrow c^* = \frac{83}{3} = 27.67$

all nonnegative

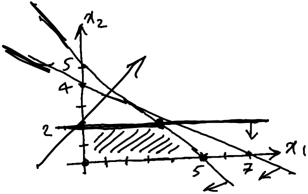
Subproblem 2

$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& x_2 \le 2, \\
& x_1, x_2 \ge 0.
\end{aligned}
$$



Solution:

$$\boxed{c^* = 27, x_1^* = 3, x_2^* = 2.}$$



Subproblem 2

$$\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 5, \\
& 4x_1 + 7x_2 \leq 28, \\
& x_2 \leq 2, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

	x_1	x_2	s_1	s_2	s_3	RHS
s_1	1	1	1	0	0	5
s_2	4	7	0	1	0	28
s_3	0	1	0	0	1	2
c	-5	-6	0	0	0	0

min test:

$\frac{5}{1} = 5$

$\frac{28}{7} = 4$

$\frac{2}{1} = 2$

		x_1	x_2	s_1	s_2	s_3	RHS
$r_1 - r_3$	s_1	1	0	1	0	-1	3
$r_2 - 7r_3$	s_2	4	0	0	1	-7	14
r_3	x_2	0	1	0	0	1	2
$r_4 + 6r_3$	c	-5	0	0	0	6	12

min test:

$\frac{3}{1} = 3$

$\frac{14}{4} = \frac{7}{2} = 3.5$

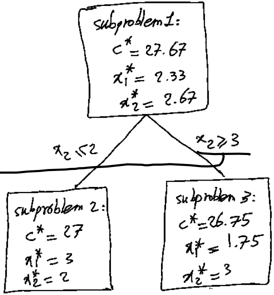
—

		x_1	x_2	s_1	s_2	s_3	RHS	
r_1	x_1	1	0	1	0	-1	3	$\Rightarrow x_1^* = 3$
$r_2 - 4r_1$	s_2	0	0	-4	1	-3	2	$\Rightarrow s_2^* = 2$
r_3	x_2	0	1	0	0	1	2	$\Rightarrow x_2^* = 2$
$r_4 + 5r_1$	c	0	0	5	0	1	27	$\Rightarrow c^* = 27$

$s_1^* = s_3^* = 0$

Subproblem 3

$$
\begin{aligned}
\max_{x_1,x_2}\quad & c = 5x_1 + 6x_2 \\
\text{subject to}\quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& \boxed{x_2 \ge 3,} \\
& x_1, x_2 \ge 0.
\end{aligned}
$$



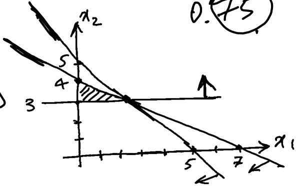
Solution:

$$c^* = 26.75, \underbrace{x_1^* = 1.75}_{0.75}, x_2^* = 3.$$

0.75

$x_2^* = .00$

$1 \le 1.75 \le 2$



Subproblem 3

$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 5, \\ & 4x_1 + 7x_2 \leq 28, \\ & x_2 \geq 3, \\ & x_1, x_2 \geq 0. \end{aligned}$$

		x_1	x_2	s_1	s_2	a_3	e_3	RHS
	s_1	1	1	1	0	0	0	5
	s_2	4	7	0	1	0	0	28
	a_3	0	1	0	0	1	-1	3
	c	-5	-6	0	0	M	0	0
$r_4 - Mr_3$	c	-5	-6-M	0	0	0	M	-3M

⋮

Subproblem 4

$$\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & \boxed{c = 5x_1 + 6x_2} \\
\text{subject to} \quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& x_2 \ge 3, \\
& x_1 \le 1, \\
& x_1, x_2 \ge 0.
\end{aligned}$$

subproblem 3:
$c^* = 26.75$
$x_1^* = 1.75$ → $1 \le 1.75 \le 2$
$x_2^* = 3$

$x_1 \le 1$ | $x_1 \ge 2$

subproblem 4:
$c^* = 25.75$
$x_1^* = 1$
$x_2^* = 3.42$

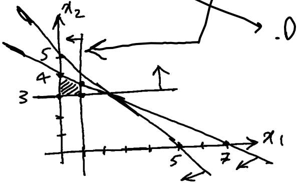
subproblem 5:
no feasible answer

Solution:

$$\boxed{c^* = 25.75, x_1^* = 1, x_2^* = 3.42.}$$

.00

0.42



Subproblem 4

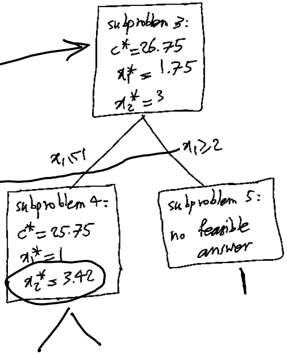
$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 5, \\
& 4x_1 + 7x_2 \leq 28, \\
& x_2 \geq 3, \\
& x_1 \leq 1, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

		x_1	x_2	s_1	s_2	s_3	a_4	e_4	RHS
	s_1	1	1	1	0	0	0	0	5
	s_2	4	7	0	1	0	0	0	28
	s_3	1	0	0	0	1	0	0	1
	a_4	0	1	0	0	0	1	-1	3
	c	-5	-6	0	0	0	M	0	0
$r_5 - Mr_4$	c	-5	$-6-M$	0	0	0	0	M	$-3M$

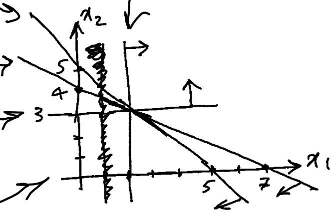
$\vdots$

Subproblem 5

$$
\begin{aligned}
\max_{x_1, x_2} \quad & \boxed{c = 5x_1 + 6x_2} \\
\text{subject to} \quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& x_2 \ge 3, \\
& x_1 \ge 2, \\
& x_1, x_2 \ge 0.
\end{aligned}
$$



Solution: <u>No feasible solution!</u>



Subproblem 6

$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \le 5, \\
& 4x_1 + 7x_2 \le 28, \\
& x_2 \ge 3, \\
& x_1 \ge 2, \\
& x_2 \le 3, \\
& x_1, x_2 \ge 0.
\end{aligned}
$$

subproblem 4: $c^* = 25.75$, $x_1^* = 1$, $x_2^* = 3.42$

$3 \le 3.42 \le 4$

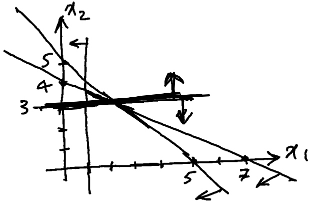
$x_2 \le 3$: subproblem 6: $c^* = 23$, $x_1^* = 1$, $x_2^* = 3$

$x_2 \ge 4$: subproblem 7: $c^* = 24$, $x_1^* = 0$, $x_2^* = 4$

Solution:

$x_2 = 3$

$$\boxed{c^* = 23, x_1^* = 1, x_2^* = 3}$$



Subproblem 7

$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 5x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 5, \\ & 4x_1 + 7x_2 \le 28, \\ & x_2 \ge 3, \\ & x_1 \ge 2, \\ & x_2 \ge 4, \\ & x_1, x_2 \ge 0. \end{aligned}$$

subproblem 4:
$c^* = 25.75$
$x_1^* = 1$
$x_2^* = 3.42$

$x_2 \le 3$

$x_2 \ge 4$

subproblem 6:
$c^* = 23$
$x_1^* = 1$
$x_2^* = 3$

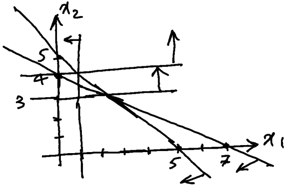
X

subproblem 7:
$c^* = 24$
$x_1^* = 0$
$x_2^* = 4$

X

Solution:

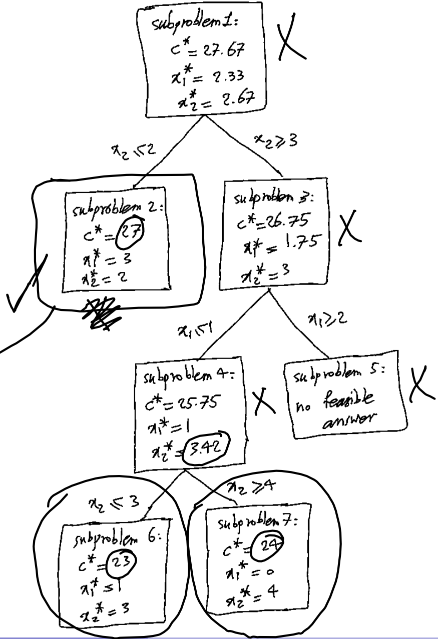
$$\boxed{c^* = 24, x_1^* = 0, x_2^* = 4}$$



Sub problems in a nutshell

- Compare solutions of subproblems 2, 5, 6, and 7 (with integer solutions).
- The final solution is the solution of subproblem 2 with the maximum objective function value:

$$\boxed{c^* = 27, x_1^* = 3, x_2^* = 2.}$$



Acknowledgment

This lecture is inspired by the lectures of Prof. Shokoufeh Mirzaei on integer linear programming: [Link]

References

[1] B. Chachuat, "Mixed-integer linear programming (MILP): Model formulation," *McMaster University Department of Chemical Engineering*, vol. 17, 2019.