# Linear Programming

Optimization Techniques (ENGG*6140)

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**Linear Programming**

# Linear programming

A **linear programming** problem is of the form:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \text{linear function in } \boldsymbol{x} \\ \text{subject to} & \text{affine inequality constraints in } \boldsymbol{x}, \\ & \text{affine equality constraints in } \boldsymbol{x}. \end{array}$$

# Standard linear programming

A **standard linear programming** problem is of the form:

Maximization:

$$\begin{aligned} \underset{\boldsymbol{x}=[x_1,\ldots,x_n]^\top}{\text{maximize}} \quad & \boldsymbol{\alpha}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \preceq \boldsymbol{h}, \\ & \boldsymbol{x} \succeq \boldsymbol{0}, \end{aligned}$$

Minimization:

$$\begin{aligned} \underset{\boldsymbol{x}=[x_1,\ldots,x_n]^\top}{\text{minimize}} \quad & \boldsymbol{\alpha}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \succeq \boldsymbol{h}, \\ & \boldsymbol{x} \succeq \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{G} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{h} \in \mathbb{R}^m$.

# Standard linear programming

Equivalently:

$$\begin{aligned}
\underset{x_1,\ldots,x_n}{\text{minimize/maximize}} \quad & \alpha_1 x_1 + \cdots + \alpha_n x_n \\
\text{subject to} \quad & \text{linear inequality constraint } 1, \\
& \vdots \\
& \text{linear inequality constraint } m, \\
& x_1, \ldots, x_n \geq 0,
\end{aligned}$$

where $m \geq n$.

For **example**:

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \geq 40, \\
& x_1 + x_2 \geq 30, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

$$\begin{aligned}
\underset{x_1,x_2}{\text{maximize}} \quad & 40x_1 + 30x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 12, \\
& 2x_1 + x_2 \leq 16, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

## Practical Examples

# Practical Example 1

- A company has two products. Let $x_1$ and $x_2$ denote the amount of the first and second products to be produced (with some scale), respectively. Therefore, $x_1, x_2 \geq 0$.
- The company has profits \$60 and \$30 for the first and second products. Therefore, the total profit of company:

$$c = \$(60x_1 + 30x_2).$$

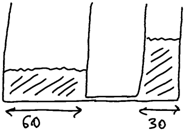
- The resources for these products are limited, so we have the following restrictions:
  - We do not want the first product, with proportion 8, and the second product, with proportion 3, to spend more than \$48, so: $8x_1 + 3x_2 \leq \$48$.
  - For four of the first product and three of the second product, we have the budget to spend at least \$25, so: $4x_1 + 2x_2 \geq \$25$.

The optimization becomes:

$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 60x_1 + 30x_2 \\ \text{subject to} \quad & 8x_1 + 3x_2 \leq 48, \\ & 4x_1 + 2x_2 \geq 25, \\ & x_1, x_2 \geq 0. \end{aligned}$$

# Practical Example 2

- We have two 2D tanks of water which are connected from their bottom. Let $x_1$ and $x_2$ denote the height of water (with some scale) in the first and second tanks, respectively. Therefore, $x_1, x_2 \geq 0$.
- The widths of the two tanks are 60 and 30 (with some scale), respectively. Therefore, the total amount of water in these tanks is $c = 60x_1 + 30x_2$.




- There are some linear physical restrictions on the amount of water poured in these tanks (because of previous tanks which water has passed to reach these tanks): $8x_1 + 3x_2 \leq 48$ and $4x_1 + 2x_2 \geq 25$.

The optimization becomes:

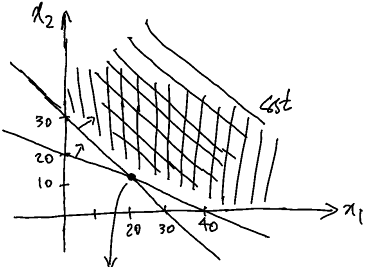
$$\begin{aligned} \underset{x_1, x_2}{\text{maximize}} \quad & c = 60x_1 + 30x_2 \\ \text{subject to} \quad & 8x_1 + 3x_2 \leq 48, \\ & 4x_1 + 2x_2 \geq 25, \\ & x_1, x_2 \geq 0. \end{aligned}$$

**Solving linear programming by visualization**

# Visualization: example 1

Minimization example:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

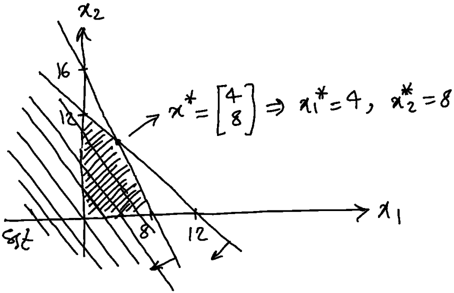



$$x^* = \begin{bmatrix} 20 \\ 10 \end{bmatrix} \Rightarrow x_1^* = 20, \; x_2^* = 10$$

# Visualization: example 2

Maximization example:

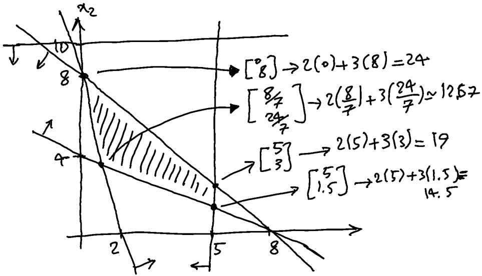
$$
\begin{aligned}
\underset{x_1, x_2}{\text{maximize}} \quad & 40x_1 + 30x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 12, \\
& 2x_1 + x_2 \leq 16, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$



# Visualization: example 3

Example with more number of constraints:

$$
\begin{array}{ll}
\underset{x_1, x_2}{\text{minimize}} & 2x_1 + 3x_2 \\
\text{subject to} & x_1 + 2x_2 \geq 8, \\
& 2x_1 + 0.5x_2 \geq 4, \\
& x_1 + x_2 \leq 8, \\
& x_1 \leq 5, \\
& x_2 \leq 10, \\
& x_1, x_2 \geq 0.
\end{array}
$$




$$x_1 + 2x_2 - 8 = 2x_1 + 0.5x_2 - 4$$

$$\hookrightarrow \quad 1.5x_2 = x_1 + 4 \rightarrow x_1 - 1.5x_2 + 4 = 0$$

$$x_1 + 2x_2 - 8 = 0$$

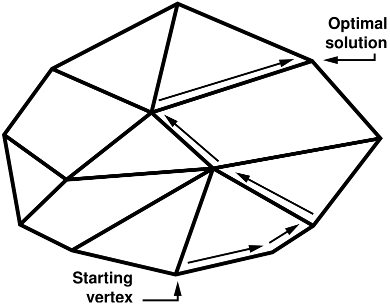
$$(1.5x_2 - 4) + 2x_2 - 8 = 0 \rightarrow 3.5x_2 = 12 \rightarrow x_2 = \frac{24}{7}$$

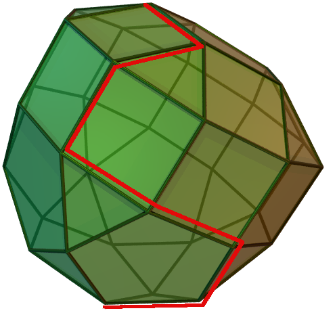
$$\hookrightarrow x_1 = 1.5x_2 - 4 = \frac{3}{\cancel{2}}\left(\frac{\overset{12}{\cancel{24}}}{7}\right) - 4 = \frac{36}{7} - 4 = \frac{8}{7}$$

## Simplex Method Description

# Simplex method description

- As you saw in the pictures, the feasible set (determined by the constraints) in the linear programming has **affine/linear boundaries**.
- It is because the constraints are affine/linear.
- Therefore, the feasible set is like a **simplex** with linear edges and some corners.
- The corners of the feasible set are named the **extreme points**.

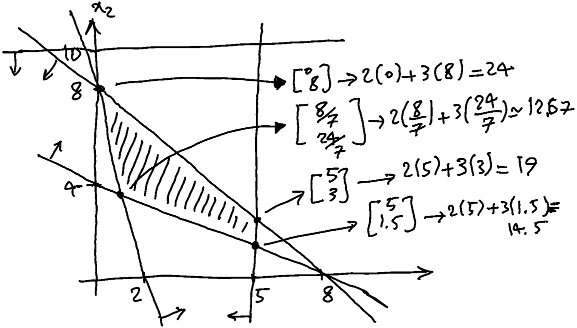


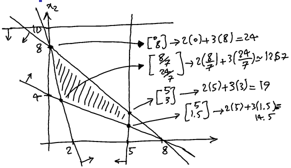
The images are taken from Wikipedia.

# Simplex method description

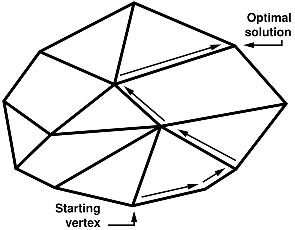
- The **simplex algorithm** was initially proposed in 1947 [1].
- It works on the **linear boundaries (edges)** and **extreme points** of the simplex feasible set.
- Obviously, the solution is at one of the **extreme points**.

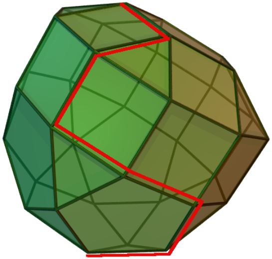


# Simplex method description




- The simplex algorithm starts from an extreme point and it goes to one of its neighbor extreme points having the smallest/largest cost function at that point (only if the neighbor extreme point has smaller/larger cost value compared to the current extreme point).
- It continues this procedure until we reach an extreme point whose neighbor extreme points do not have smaller/larger cost value.



The images are taken from Wikipedia.

**One of the methods for Simplex Algorithm: Tableau Method for Maximization**

# Slack variables

Consider this example:

$$\begin{aligned} \underset{x_1, x_2, x_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \leq 240, \\ & x_1 + 3x_2 + 2x_3 \leq 360, \\ & 2x_1 + x_2 + 2x_3 \leq 300, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

- We convert each inequality $\leq$ constraint to an equality constraint by adding **slack variables**.
- Slack variables are **non-negative scalars** which are added to the left hand side of inequality $\leq$ constraint to make it equality.
- Example:

$$\begin{aligned} 2x_1 + x_2 + x_3 3 \leq 240 &\implies 2x_1 + x_2 + x_3 + s_1 = 240, \\ x_1 + 3x_2 + 2x_3 \leq 360 &\implies x_1 + 3x_2 + 2x_3 + s_2 = 360, \\ 2x_1 + x_2 + 2x_3 \leq 300 &\implies 2x_1 + x_2 + 2x_3 + s_3 = 300, \\ s_1, s_2, s_3 \geq 0. \end{aligned}$$

# Slack variables

So, this problem:

$$\begin{aligned}
\underset{x_1, x_2, x_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 \leq 240, \\
& x_1 + 3x_2 + 2x_3 \leq 360, \\
& 2x_1 + x_2 + 2x_3 \leq 300, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

is converted to:

$$\begin{aligned}
\underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

# Forming equalities

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & 6x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The cost function is: $c := 6x_1 + 5x_2 + 4x_3 \implies c - 6x_1 - 5x_2 - 4x_3 = 0$.
Therefore:

$$
\begin{aligned}
& 2x_1 + x_2 + x_3 + s_1 = 240, \\
& x_1 + 3x_2 + 2x_3 + s_2 = 360, \\
& 2x_1 + x_2 + 2x_3 + s_3 = 300, \\
& c - 6x_1 - 5x_2 - 4x_3 = 0.
\end{aligned}
$$

# Forming the table in the tableau method

$$
\begin{aligned}
&2x_1 + x_2 + x_3 + s_1 = 240,\\
&x_1 + 3x_2 + 2x_3 + s_2 = 360,\\
&2x_1 + x_2 + 2x_3 + s_3 = 300,\\
&c - 6x_1 - 5x_2 - 4x_3 = 0.
\end{aligned}
$$

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 1 | 0 | 0 | 240 |
| $s_2$ | 1 | 3 | 2 | 0 | 1 | 0 | 360 |
| $s_3$ | 2 | 1 | 2 | 0 | 0 | 1 | 300 |
| $c$ | -6 | -5 | -4 | 0 | 0 | 0 | 0 |

# Pivot and min test

1. In **maximization** problem, choose the **most negative** value, in the row of cost, for the **pivot column**.
2. Do the **min test**: divide RHS values (of rows except the c row) to the values of the pivot column. **Ignore** the negative or zero values in min test.
3. Get the minimum division value for the **pivot row**. The intersection of pivot row and pivot value gives the **pivot value**.

pivot row

pivot value

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 1 | 0 | 0 | 240 |
| $s_2$ | 1 | 3 | 2 | 0 | 1 | 0 | 360 |
| $s_3$ | 2 | 1 | 2 | 0 | 0 | 1 | 300 |
| c | -6 | -5 | -4 | 0 | 0 | 0 | 0 |

min test

$$\frac{240}{2} = 120$$

$$\frac{360}{1} = 360$$

$$\frac{300}{2} = 150$$

most negative value in c row

pivot column

# Simplifying the pivot column

1. Make the pivot value **one** and other values **zero** in the pivot column.
2. For every row, use **the row itself** and the **pivot row** only.
3. **Replace the name** of the pivot row with the name of the pivot column.

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 1 | 0 | 0 | 240 |
| $s_2$ | 1 | 3 | 2 | 0 | 1 | 0 | 360 |
| $s_3$ | 2 | 1 | 2 | 0 | 0 | 1 | 300 |
| $c$ | -6 | -5 | -4 | 0 | 0 | 0 | 0 |

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $r_1/2$ | $x_1$ | 1 | 0.5 | 0.5 | 0.5 | 0 | 0 | 120 |
| $r_2 - \frac{r_1}{2}$ | $s_2$ | 0 | 2.5 | 1.5 | -0.5 | 1 | 0 | 240 |
| $r_3 - r_2$ | $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 |
| $r_4 + 3r_1$ | $c$ | 0 | -2 | -1 | 3 | 0 | 0 | 720 |

# Continuing the table

- In the **maximization** problem, we continue the table until all the values in the c row are non-negative (positive or zero).

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 1 | 0 | 0 | 240 |
| $s_2$ | 1 | 3 | 2 | 0 | 1 | 0 | 360 |
| $s_3$ | 2 | 1 | 2 | 0 | 0 | 1 | 300 |
| c | -6 | -5 | -4 | 0 | 0 | 0 | 0 |

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $r_1/2$ | $x_1$ | 1 | 0.5 | 0.5 | 0.5 | 0 | 0 | 120 |
| $r_2 - \frac{r_1}{2}$ | $s_2$ | 0 | 2.5 | 1.5 | -0.5 | 1 | 0 | 240 |
| $r_3 - r_2$ | $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 |
| $r_4 + 3r_1$ | C | 0 | -2 | -1 | 3 | 0 | 0 | 720 |

# Continuing the table

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS | min test |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0.5 | 0.5 | 0.5 | 0 | 0 | 120 | $\frac{120}{0.5} = 240$ |
| $s_2$ | 0 | 2.5 | 1.5 | -0.5 | 1 | 0 | 240 | $\frac{240}{2.5} = 96$ |
| $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 | — |
| $C$ | 0 | -2 | -1 | 3 | 0 | 0 | 720 | |

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $r_1 - \frac{r_2}{5}$ | $x_1$ | 1 | 0 | $1/5$ | $3/5$ | $-1/5$ | 0 | 72 |
| $r_2/2.5$ | $x_2$ | 0 | 1 | $3/5$ | $-1/5$ | $2/5$ | 0 | 96 |
| $r_3$ | $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 |
| $r_4 + \frac{4}{5} r_2$ | $C$ | 0 | 0 | $\frac{1}{5}$ | $\frac{13}{5}$ | $\frac{4}{5}$ | 0 | 912 |

all non-negative

maximum cost function ($c^*$)

# Basic and non-basic variables

Once the **table is over**:

- A column with having only one 1 and the rest 0 is a **basic variable**.
- The other columns are **non-basic variables**.

non-basic variables

basic variables

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $r_1 - \frac{r_2}{5}$ | $x_1$ | 1 | 0 | 1/5 | 3/5 | -1/5 | 0 | 72 |
| $r_2/2.5$ | $x_2$ | 0 | 1 | 3/5 | -1/5 | 2/5 | 0 | 96 |
| $r_3$ | $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 |
| $r_4 + \frac{4}{5} r_2$ | $c$ | 0 | 0 | 1/5 | 13/5 | 4/5 | 0 | 912 |

all non-negative
(table over)

# Checking the optimal values

- Once the table is over, the RHS of the c row is the **optimal cost function**. Here it is $c^* = 912$.
- The optimal values for the variables are the RHS of the rows. In other words, the optimum basic variables are the RHS of rows. Here they are $x_1^* = 72, x_2^* = 96, s_3^* = 60$.
- The optimum value for the rest of the variables (the **non-basic variables**) is **zero**. Here they are $x_3^* = 0, s_1^* = 0, s_2^* = 0$.
- We can check if the optimal cost is correct:

$$c := 6x_1 + 5x_2 + 4x_3 \implies c^* = 6x_1^* + 5x_2^* + 4x_3^* = 6(72) + 5(96) + 4(0) = 912 \quad \checkmark$$

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $r_1 - \frac{r_2}{5}$ | $x_1$ | 1 | 0 | 1/5 | 3/5 | -1/5 | 0 | 72 |
| $r_2/2.5$ | $x_2$ | 0 | 1 | 3/5 | -1/5 | 2/5 | 0 | 96 |
| $r_3$ | $s_3$ | 0 | 0 | 1 | -1 | 0 | 1 | 60 |
| $r_4 + \frac{4}{5} r_2$ | c | 0 | 0 | 1/5 | 13/5 | 4/5 | 0 | 912 |

all non-negative

maximum cost function ($c^*$)

**Big M method**

# When to use the big M method

We should use the big M method when there are one or some $\geq$ constraints and/or $=$ constraints. In other words, whenever we have **mixed constraints**.

Consider this example with $\leq$ and $\geq$ constraints:

$$\begin{aligned}
\underset{x_1,x_2}{\text{maximize}} \quad & c = 3x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 600, \\
& x_1 + x_2 \leq 225, \\
& 5x_1 + 4x_2 \leq 1000, \\
& x_1 + 2x_2 \geq 150, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

- For $\leq$ constraints, we use slack variables as before:

$$\begin{aligned}
& 2x_1 + x_2 \leq 600 \implies 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 \leq 225 \implies x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 \leq 1000 \implies 5x_1 + 4x_2 + s_3 = 1000, \\
& s_1, s_2, s_3 \geq 0.
\end{aligned}$$

# Big M method: $\geq$ constraints

$$\begin{aligned} \underset{x_1,x_2}{\text{maximize}} \quad & c = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 600, \\ & x_1 + x_2 \leq 225, \\ & 5x_1 + 4x_2 \leq 1000, \\ & x_1 + 2x_2 \geq 150, \\ & x_1, x_2 \geq 0. \end{aligned}$$

- For $\geq$ constraints, we can't use slack variables because the slack variable will not be non-negative anymore:

$$x_1 + 2x_2 \geq 150 \implies x_1 + 2x_2 + s_4 = 150 \implies s_4 \leq 0.$$

- For $\geq$ constraints, we use excess variables $e$ and artificial variables $a$:

$$\begin{aligned} & x_1 + 2x_2 \geq 150 \implies x_1 + 2x_2 + a_4 - e_4 = 150, \\ & a_4, e_4 \geq 0. \end{aligned}$$

- We want the additional variable to be very small ($a_4 = \epsilon$) so we add it to the cost function with a very big multiplication factor $M \gg 1$:

$$\underset{x_1,x_2,x_3}{\text{maximize}} \quad c = 3x_1 + 4x_2 - Ma_4,$$

because if $M \gg 1$, then $a_4 \to 0$ to cancel its effect in the cost function.

# Tableau method with the big M method

$$
\begin{aligned}
\underset{x_1, x_2, s_1, s_2, s_3, a_4, e_4}{\text{maximize}} \quad & c = 3x_1 + 4x_2 - Ma_4 \\
\text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 + s_3 = 1000, \\
& x_1 + 2x_2 + a_4 - e_4 = 150, \\
& x_1, x_2, s_1, s_2, s_3, a_4, e_4 \geq 0.
\end{aligned}
$$

- We make zero the column value of additional variable in the c row, because the value of $a_4$ should be about zero rather than $M$.

| | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $a_4$ | $e_4$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| | $s_1$ | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 600 |
| | $s_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 225 |
| | $s_3$ | 5 | 4 | 0 | 0 | 1 | 0 | 0 | 1000 |
| | $a_4$ | 1 | 2 | 0 | 0 | 0 | 1 | -1 | 150 |
| | $c$ | -3 | -4 | 0 | 0 | 0 | $M$ | 0 | |
| $r_5 - Mr_4$ | $c$ | $-3-M$ | $-2M-4$ | 0 | 0 | 0 | 0 | $M$ | $-150M$ |

# Tableau method with the big M method

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $a_4$ | $e_4$ | RHS | min test |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 600 | $\frac{600}{1} = 600$ |
| $s_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 225 | $\frac{225}{1} = 225$ |
| $s_3$ | 5 | 4 | 0 | 0 | 1 | 0 | 0 | 1000 | $\frac{1000}{4} = 250$ |
| $a_4$ | 1 | 2 | 0 | 0 | 0 | 1 | $-1$ | 150 | $\frac{150}{2} = 75$ |
| $C$ | $-3$ | $-4$ | 0 | 0 | 0 | $M$ | 0 | | |
| $C$ | $-3-M$ | $-2M-4$ | 0 | 0 | 0 | 0 | $M$ | $-150M$ | |

| | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $a_4$ | $e_4$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $r_1 - \frac{r_4}{2}$ | $s_1$ | $3/2$ | 0 | 1 | 0 | 0 | $-1/2$ | $1/2$ | 575 |
| $r_2 - r_4/2$ | $s_2$ | $\frac{1}{2}$ | 0 | 0 | 1 | 0 | $-1/2$ | $1/2$ | 150 |
| $r_3 - 2r_4$ | $s_3$ | 3 | 0 | 0 | 0 | 1 | $-2$ | 2 | 700 |
| $r_4/2$ | $x_2$ | $\frac{1}{2}$ | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{-1}{2}$ | 75 |
| | $C$ | $-1$ | 0 | 0 | 0 | 0 | $M+2$ | $-2$ | 300 |

$r_6 + \left(\frac{2M+4}{2}\right) r_4 = r_6 + (M+2) r_4$

# Tableau method with the big M method

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $a_4$ | $e_4$ | RHS | min test |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | $\frac{3}{2}$ | 0 | 1 | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 575 | $\frac{575}{0.5} = 1150$ |
| $s_2$ | $\frac{1}{2}$ | 0 | 0 | 1 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 150 | $\frac{150}{0.5} = 300$ |
| $s_3$ | 3 | 0 | 0 | 0 | 1 | $-2$ | 2 | 700 | $\frac{700}{2} = 350$ |
| $x_2$ | $\frac{1}{2}$ | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 75 | — |
| $c$ | $-1$ | 0 | 0 | 0 | 0 | $M+2$ | $-2$ | 300 | |

| | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $a_4$ | $e_4$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $r_1 - r_2$ | $s_1$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 375 |
| $2r_2$ | $e_4$ | 1 | 0 | 0 | 2 | 0 | $-1$ | 1 | 300 |
| $r_3 - 4r_2$ | $s_3$ | 1 | 0 | 0 | $-4$ | 1 | 0 | 0 | 100 |
| $r_4 + r_2$ | $x_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 225 |
| $r_5 + 4r_2$ | $c$ | 1 | 0 | 0 | 4 | 0 | $M$ | 0 | 900 |

$\underbrace{1 \quad 0 \quad 0 \quad 4 \quad 0 \quad M \quad 0}_{\text{all positive}}$

Therefore: $s_1^* = 375, e_4^* = 300, s_3^* = 100, x_2^* = 225, x_1^* = 0, s_2^* = 0, s_3^* = 0, a_4^* = 0, c^* = 900.$
Check: $c^* = 3x_1^* + 4x_2^* = 3(0) + 4(225) = 900 \quad \surd$

# Example 2 for mixed constraints

Consider another example with mixed constraints:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}} \quad & c = x_1 - x_2 + 3x_3 + 4 \\ \text{subject to} \quad & x_1 + x_2 \leq 20, \\ & x_1 + x_3 = 5, \\ & x_2 + x_3 \geq 10, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

- We drop the DC value from the cost for now:

$$c = x_1 - x_2 + 3x_3.$$

- We have:

$$\begin{aligned} & x_1 + x_2 \leq 20 \implies x_1 + x_2 + s_1 = 20, \\ & x_1 + x_3 = 5 \implies x_2 + x_3 + a_1 = 5, \\ & x_2 + x_3 \geq 10 \implies x_2 + x_3 + a_2 + e_2 = 10, \\ & s_1, a_1, a_2, e_2 \geq 0. \end{aligned}$$

# Example 2 for mixed constraints

The problem is converted to:

$$
\begin{aligned}
\underset{x_1, x_2, x_3}{\text{maximize}} \quad & c = x_1 - x_2 + 3x_3 - Ma_1 - Ma_2 \\
\text{subject to} \quad & x_1 + x_2 + s_1 = 20, \\
& x_2 + x_3 + a_1 = 5, \\
& x_2 + x_3 + a_2 + e_2 = 10, \\
& s_1, a_1, a_2, e_2 \geq 0.
\end{aligned}
$$

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $a_1$ | $a_2$ | $e_2$ | RHS | min test |
|---|---|---|---|---|---|---|---|---|---|---|
| | $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 20 | — |
| | $a_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 5 | $5/1=5$ |
| | $a_2$ | 0 | 1 | 1 | 0 | 0 | 1 | $-1$ | 10 | $10/1=10$ |
| | $c$ | $-1$ | $+1$ | $-3$ | 0 | $+M$ | $+M$ | 0 | 0 | |
| $r_4 - Mr_2$ | $c$ | $-M-1$ | 1 | $-M-3$ | 0 | 0 | $M$ | 0 | $-5M$ | |
| $r_5 - Mr_3$ | $c$ | $-M-1$ | $-M+1$ | $-2M-3$ | 0 | 0 | 0 | $M$ | $-15M$ | |

# Example 2 for mixed constraints

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $a_1$ | $a_2$ | $e_2$ | RHS | min test: |
|---|---|---|---|---|---|---|---|---|---|---|
| | $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 20 | — |
| | $a_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 5 | $5/1=5$ |
| | $a_2$ | 0 | 1 | 1 | 0 | 0 | 1 | $-1$ | 10 | $10/1=10$ |
| | $c$ | $-1$ | $+1$ | $-3$ | 0 | $+M$ | $+M$ | 0 | 0 | |
| $r_4-Mr_2$ | $c$ | $-M-1$ | 1 | $-M-3$ | 0 | 0 | $M$ | 0 | $-5M$ | |
| $r_5-Mr_3$ | $c$ | $-M-1$ | $-M+1$ | $-2M-3$ | 0 | 0 | 0 | $M$ | $-15M$ | |

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $a_1$ | $a_2$ | $e_2$ | RHS | min test: |
|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 20 | $20/1=20$ |
| $r_2$ | $x_3$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 5 | — |
| $r_3-r_2$ | $a_2$ | $-1$ | 1 | 0 | 0 | $-1$ | 1 | $-1$ | 5 | $\frac{5}{1}=5$ |
| $r_6+(2M+3)r_2$ | $c$ | $M+2$ | $-M+1$ | 0 | 0 | $2M+3$ | 0 | $M$ | $-5M+15$ | |

# Example 2 for mixed constraints

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $a_1$ | $a_2$ | $e_2$ | RHS | min test: |
|---|---|---|---|---|---|---|---|---|---|---|
| $r_1$ | $s_1$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 20 | $20/1 = 20$ |
| $r_2$ | $x_3$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 5 | — |
| $r_3 - r_2$ | $a_2$ | $-1$ | 1 | 0 | 0 | $-1$ | 1 | $-1$ | 5 | $\frac{5}{1} = 5$ |
| $r_6 + (2M+3)r_2$ | $c$ | $M+2$ | $-M+1$ | 0 | 0 | $2M+3$ | 0 | $M$ | $-5M+15$ | |

| | | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $a_1$ | $a_2$ | $e_2$ | RHS | |
|---|---|---|---|---|---|---|---|---|---|---|
| $r_1 - r_3$ | $s_1$ | 2 | 0 | 0 | 1 | 1 | $-1$ | 1 | 15 | $\Rightarrow s_1^* = 15$ |
| $r_2$ | $x_3$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 5 | $\Rightarrow x_3^* = 5$ |
| $r_3$ | $x_2$ | $-1$ | 1 | 0 | 0 | $-1$ | 1 | $-1$ | 5 | $\Rightarrow x_2^* = 5$ |
| $r_4 + (M-1)r_3$ | $c$ | 3 | 0 | 0 | 0 | $M+4$ | $M-1$ | 1 | 10 | $\Rightarrow c^* = 10$ |

all non-negative!

Therefore: $s_1^* = 15, x_3^* = 5, x_2^* = 5, c^* = 10, x_1^* = a_1^* = a_2^* = e_2^* = 0$.
Check: $c^* = x_1^* - x_2^* + 3x_3^* = 0 - 5 + 3(5) = 10$ $\surd$
The final answer for maximum actual cost is (we add back the DC value): $c^* = 10 + 4 = 14$.

## The Reason for the Tableau Method

# The reason for the tableau method

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 \leq 50, \\
& 2x_1 + 3x_2 + x_4 \leq 42, \\
& 3x_3 - x_4 \leq 250, \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

is converted to:

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4,s_1,s_2,s_3}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + s_1 = 50, \\
& 2x_1 + 3x_2 + x_4 + s_2 = 42, \\
& 3x_3 - x_4 + s_3 = 250, \\
& x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

- $\#$ variables: 7, $\#$ equations: 3
- We can set $7 - 3 = 4$ variables to zero (non-basic variables) and find the other 3 variables (basic variables).
- How many ways can we choose the three variables out of the 7 variables? $\binom{7}{3} = 35$.

# Example variables to choose

One of the ways:

$$\text{non-basic variables: } x_1 = x_2 = x_3 = x_4 = 0,$$
$$\text{basic variables: } s_1, s_2, s_3.$$

$$\begin{aligned}
\underset{s_1, s_2, s_3}{\text{maximize}} \quad & c = 0 \\
\text{subject to} \quad & s_1 = 50, \\
& s_2 = 42, \\
& s_3 = 250, \\
& s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Therefore, $s_1 = 50, s_2 = 42, s_3 = 250$.
The cost function becomes: $c = 0$.

# Example variables to choose

One of the ways:

non-basic variables: $x_1 = x_4 = s_1 = s_2 = 0$,
basic variables: $x_2, x_3, s_3$.

$$
\begin{aligned}
\underset{x_2, x_3, s_3}{\text{maximize}} \quad & c = 6x_2 \\
\text{subject to} \quad & x_2 + x_3 = 50, \\
& 3x_2 = 42, \\
& 3x_3 + s_3 = 250, \\
& x_2, x_3, s_3 \geq 0.
\end{aligned}
$$

Therefore, $x_2 = 14, x_3 = 36, s_3 = 142$.
The cost function becomes: $c = 6(14) = 84$.

# The reason for the pivot column

Which variable should we increase which maximizes the cost function the most?

$$c = 4x_1 + 6x_2 - 5x_4.$$

Increasing the variable $x_2$ has the most effect because it has the biggest multiplication factor, i.e., 6.
Recall that we had:

$$c - 4x_1 - 6x_2 + 5x_4 = 0.$$

That is why, in the tableau method, we find the most negative value in the $c$ row. This is the reason for the **pivot column**.

# The reason for the min test

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4,s_1,s_2,s_3}{\text{maximize}} \quad & c = 4x_1 + 6x_2 - 5x_4 \\
\text{subject to} \quad & x_1 + x_2 + x_3 + s_1 = 50, \\
& 2x_1 + 3x_2 + x_4 + s_2 = 42, \\
& 3x_3 - x_4 + s_3 = 250, \\
& x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | RHS | min test |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 50 | $\frac{50}{1} = 50$ |
| $s_2$ | 2 | 3 | 0 | 1 | 0 | 1 | 0 | 42 | $\frac{42}{3} = 14$ |
| $s_3$ | 0 | 0 | 3 | -1 | 0 | 0 | 1 | 250 | — |
| $c$ | -4 | -6 | 0 | 5 | 0 | 0 | 0 | 0 | |

How much can we increase the $x_2$ variable?

- In the first constraint, the worst case scenario is $x_1 = x_3 = s_1 = 0$ and the most we can increase $x_2$: $x_2 = 50$
- In the second constraint, the worst case scenario is $x_1 = x_4 = s_2 = 0$ and the most we can increase $x_2$: $3x_2 = 42 \implies x_2 = 42/3 = 14$
- In the third constraint, the worst case scenario is $x_3 = x_4 = s_3 = 0$ and the most we can increase $x_2$: $30x_2 = 250 \implies x_2 = \infty$
- Therefore, the minimum increase we can have for $x_2$ is: $\min(50, 42, \infty) = 42$.

## Solving the Dual Problem for Minimization

# Dual problem for minimization

An example **minimization** linear problem is:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

- When we have a minimization linear programming, we can **convert** the minimization problem to a maximization problem.
- We should find the **dual problem** for the minimization problem. The dual for the minimization is a maximization problem. We will learn the dual problem of linear programming soon.

# Dual problem for minimization

An example **minimization** linear problem is:

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 40, \\ & x_1 + x_2 \geq 30, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Consider the constraints:

$$x_1 + 2x_2 \geq 40 \overset{\times y_1}{\Longrightarrow} y_1 x_1 + 2y_1 x_2 \geq 40 y_1,$$
$$x_1 + x_2 \geq 30 \overset{\times y_2}{\Longrightarrow} y_2 x_1 + y_2 x_2 \geq 30 y_2,$$

where $y_1, y_2 \geq 0$. Summing the sides together gives:

$$(y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

On the other hand, the cost of dual problem is a lower bound on the cost of the primal problem:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2.$$

# Dual problem for minimization

Summing the sides together gives:

$$(y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

On the other hand, the cost of dual problem is a lower bound on the cost of the primal problem:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2.$$

Therefore:

$$12x_1 + 16x_2 \geq (y_1 + y_2)x_1 + (2y_1 + y_2)x_2 \geq 40y_1 + 30y_2.$$

Hence:

$$\begin{aligned} y_1 + y_2 &\leq 12, \\ 2y_1 + y_2 &\leq 16. \end{aligned}$$

We want to find the best (maximum) lower bound, so:

$$\underset{y_1, y_2}{\text{maximize}} \quad 40y_1 + 30y_2.$$

# Dual problem for minimization

Therefore:

$$\begin{aligned}
\underset{y_1, y_2}{\text{maximize}} \quad & 40y_1 + 30y_2 \\
\text{subject to} \quad & y_1 + y_2 \leq 12, \\
& 2y_1 + y_2 \leq 16, \\
& y_1, y_2 \geq 0.
\end{aligned}$$

is the dual problem for the following problem:

$$\begin{aligned}
\underset{x_1, x_2}{\text{minimize}} \quad & 12x_1 + 16x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \geq 40, \\
& x_1 + x_2 \geq 30, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

This maximization problem can be solved as explained before.

# Solving the problem by tableau method

$$
\begin{aligned}
\underset{y_1, y_2}{\text{maximize}} \quad & c = 40y_1 + 30y_2 \\
\text{subject to} \quad & y_1 + y_2 + s_1 = 12, \\
& 2y_1 + y_2 + s_2 = 16, \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

| | $y_1$ | $y_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 12 |
| $s_2$ | 2 | 1 | 0 | 1 | 16 |
| $c$ | -40 | -30 | 0 | 0 | 0 |

$\underbrace{\text{min test}}$

$12/1 = 12$

$16/2 = 8$

| | | $y_1$ | $y_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|---|
| $r_1 - \frac{r_2}{2}$ | $s_1$ | 0 | 0.5 | 1 | -0.5 | 4 |
| $\frac{r_2}{2}$ | $y_1$ | 1 | 0.5 | 0 | 0.5 | 8 |
| $r_3 + 20r_2$ | $c$ | 0 | -10 | 0 | 20 | 320 |

# Solving the problem by tableau method

| | $y_1$ | $y_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| $s_1$ | 0 | 0.5 | 1 | -0.5 | 4 |
| $y_1$ | 1 | 0.5 | 0 | 0.5 | 8 |
| $c$ | 0 | -10 | 0 | 20 | 320 |

$\underbrace{\text{min test}}$

$4/0.5 = 8$

$8/0.5 = 16$

| | | $y_1$ | $y_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|---|
| $2r_1$ | $y_2$ | 0 | 1 | 2 | -1 | 8 |
| $r_2 - r_1$ | $y_1$ | 1 | 0 | -1 | 1 | 4 |
| $r_3 + 20r_1$ | $c$ | 0 | 0 | 20 | 10 | 400 |

all positive

Therefore: $y_2^* = 8, y_1^* = 4, s_1^* = 0, s_2^* = 0, c^* = 400.$
Check: $c^* = 40y_1^* + 30y_2^* = 40(4) + 30(8) = 400 \quad \surd$

**The strong duality holds for linear programming**, so:

$$c^* = 400 \text{ for the primal problem, too.}$$

## Dual Simplex Method

# Why we need the dual simplex method?

- We converted the minimization linear problem to its **dual problem** which is the maximization linear problem. Then, we solved it using the simplex method for maximization.
- However, it only gave us the **optimal cost function** $c^*$ and not the **optimum primal variables** $\{x_1^*, \ldots, x_n^*\}$.
- For finding these optimum primal variables in the minimization linear programming, we can use the **dual simplex method**.
- The dual simplex method only works for the **minimization** linear problem if:
  - **all** its multiplication factors in the **cost** function are **non-negative**.
  - **at least one** of the **inequality constraints** is $\geq$.

# Dual simplex method: example

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimize}} \quad & c = 3x_1 + 4x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \le 600, \\
& x_1 + x_2 \le 225, \\
& 5x_1 + 4x_2 \le 1000, \\
& x_1 + 2x_2 \ge 150, \\
& x_1, x_2 \ge 0.
\end{aligned}$$

For inequality $\ge$, we have:

$$x_1 + 2x_2 \ge 150 \implies -x_1 - 2x_2 \le -150$$

Using slack variables:

$$\begin{aligned}
\underset{x_1,x_2}{\text{minimize}} \quad & c - 3x_1 + 4x_2 = 0 \\
\text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 + s_3 = 1000, \\
& -x_1 - 2x_2 + s_4 = -150, \\
& x_1, x_2 \ge 0.
\end{aligned}$$

# Dual simplex method: example

$$\begin{aligned}
\underset{x_1, x_2}{\text{minimize}} \quad & c - 3x_1 + 4x_2 = 0 \\
\text{subject to} \quad & 2x_1 + x_2 + s_1 = 600, \\
& x_1 + x_2 + s_2 = 225, \\
& 5x_1 + 4x_2 + s_3 = 1000, \\
& -x_1 - 2x_2 + s_4 = -150, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

1. Pivot row: Pick the most negative value in RHS
2. min test: Divide the non-zero values of c row by the negative values of the pivot row. Take absolute value in division.

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 0 | 0 | 0 | 600 |
| $s_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 225 |
| $s_3$ | 5 | 4 | 0 | 0 | 1 | 0 | 1000 |
| $s_4$ | -1 | -2 | 0 | 0 | 0 | 1 | -150 |
| $c$ | -3 | -4 | 0 | 0 | 0 | 0 | 0 |

min test: $\begin{cases} \left|\frac{-3}{-1}\right| = 3 \\ \left|\frac{-4}{-2}\right| = 2 \end{cases}$

# Dual simplex method: example

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | 1 | 1 | 0 | 0 | 0 | 600 |
| $s_2$ | 1 | 1 | 0 | 1 | 0 | 0 | 225 |
| $s_3$ | 5 | 4 | 0 | 0 | 1 | 0 | 1000 |
| $s_4$ | -1 | -2 | 0 | 0 | 0 | 1 | -150 |
| $c$ | -3 | -4 | 0 | 0 | 0 | 0 | 0 |

min test: $\begin{cases} \left|\frac{-3}{-1}\right| = 3 \\ \left|\frac{-4}{-2}\right| = 2 \end{cases}$

| | | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | $s_1$ | 1.5 | 0 | 1 | 0 | 0 | 0.5 | 525 |
| $r_2 + \frac{r_4}{2}$ | $s_2$ | 0.5 | 0 | 0 | 1 | 0 | 0.5 | 150 |
| $r_3 + 2r_4$ | $s_3$ | 3 | 0 | 0 | 0 | 1 | 2 | 700 |
| $\frac{-r_4}{2}$ | $x_2$ | 0.5 | 1 | 0 | 0 | 0 | -0.5 | 75 |
| | $c$ | -1 | 0 | 0 | 0 | 0 | -2 | 300 |

all positive

Therefore: $s_1^* = 525, s_2^* = 150, s_3^* = 700, x_2^* = 75, c^* = 300, x_1^* = 0, s_4^* = 0.$
Check: $c^* = 3x_1^* + 4x_2^* = 3(0) + 4(75) = 300$ $\checkmark$

# Dual simplex method for $\geq$ constraints in maximization

We can also use the **dual simplex method** for handling $\geq$ **constraints** in **maximization**. Example:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leq 48, \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8, \\ & x_2 \geq 1, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

We can convert the $\geq$ constraints to $\leq$ constraints by **multiplying the sides of inequality by** $-1$:

$$x_2 \geq 1 \implies -x_2 \leq -1 \implies -x_2 + s_4 = -1.$$

So, the problem is converted to:

$$\begin{aligned} \underset{x_1,x_2,x_3,s_1,s_2,s_3,s_4}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & -x_2 + s_4 = -1, \\ & x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0. \end{aligned}$$

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$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3,s_4}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& -x_2 + s_4 = -1, \\
& x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0.
\end{aligned}
$$

| | $x_2$ | $s_2$ | $s_3$ | $s_1$ | $x_3$ | $x_1$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | $-2$ | 2 | $-8$ | 1 | 0 | 0 | 0 | 24 |
| $x_3$ | $-2$ | 2 | $-4$ | 0 | 1 | 0 | 0 | 8 |
| $x_1$ | $\frac{5}{4}$ | $-\frac{1}{2}$ | $\frac{3}{2}$ | 0 | 0 | 1 | 0 | 2 |
| $s_4$ | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| $c$ | 5 | 10 | 10 | 0 | 0 | 0 | 0 | 280 |

| | $x_2$ | $s_2$ | $s_3$ | $s_1$ | $x_3$ | $x_1$ | $s_4$ | RHS |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | 2 | $-8$ | 1 | 0 | 0 | $-2$ | 26 |
| $x_3$ | 0 | 2 | $-4$ | 0 | 1 | 0 | $-2$ | 10 |
| $x_1$ | 0 | $-0.5$ | 1.5 | 0 | 0 | 1 | 1.25 | 0.75 |
| $x_2$ | 1 | 0 | 0 | 0 | 0 | 0 | $-1$ | 1 |
| $c$ | 10 | 10 | 0 | 0 | 0 | 0 | 5 | 275 |

all nonnegative

$s_1^* = 26, x_3^* = 10, x_1^* = 0.75, x_2^* = 1, c^* = 275$

# Acknowledgment

This lecture is inspired by the lectures of Prof. Shokoufeh Mirzaei on linear programming: [Link]

# References

[1] G. B. Dantzig, "Reminiscences about the origins of linear programming," in *Mathematical Programming The State of the Art*, pp. 78–86, Springer, 1983.