Sensitivity Analysis in Linear Programming

Optimization Techniques (ENGG*6140)

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Matrices in Linear Programming

Matrices in linear programming

Consider n variables and m constraints (excluding the constraints for $\underbrace{x_1, \ldots, x_n \geq 0}$).
After having slack variables, we can have:

$$
\begin{array}{lll}
\underset{x_1,\ldots,x_n}{\text{maximize}} & c = c_1 x_1 + \cdots + c_n x_n & c_j x_j \\
\text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n = b_1, & a_{ij} \\
& a_{21}x_1 + \cdots + a_{2n}x_n = b_2, & \\
& \vdots & x_j \text{ in } i\text{-th constraint} \\
& a_{m1}x_1 + \cdots + a_{mn}x_n = b_m, & \\
& \underbrace{x_1, \ldots, x_n \geq 0.} &
\end{array}
$$

$b_i \rightarrow$ RHS of ith constraint

Matrices in linear programming

Example:

$$\begin{aligned} \underset{x_1, x_2, x_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leq 48, \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

It is converted to:

$$\begin{aligned} \underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

Matrices in linear programming

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Assume we solve it until the end and at the end, the basic variables are s_1, x_3, x_1 and the non-basic variables are x_2, s_2, s_3.

- basic and non-basic variables:

define

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top$$

- the coefficients of basic and non-basic variables in the objective function:

$$\boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top$$

- the coefficients of the variables in the constraints:

$$\begin{aligned}
&\boldsymbol{a}_{x_1} := [8, 4, 2]^\top, \boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top, \boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top, \\
&\boldsymbol{a}_{s_1} := [1, 0, 0]^\top, \boldsymbol{a}_{s_2} := [0, 1, 0]^\top, \boldsymbol{a}_{s_3} := [0, 0, 1]^\top, \\
&\boldsymbol{b} := [48, 20, 8]^\top
\end{aligned}$$

Matrices in linear programming

$c_b^T x_b + c_n^T x_n^{=0}$

$n = 6$

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

$m = 3$

b

- basic and non-basic variables:

$B x_b = b$

$x_b = B^{-1} b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top$$

- the coefficients of the variables in the constraints:

$$\boldsymbol{a}_{x_1} := [8, 4, 2]^\top, \boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top, \boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top,$$
$$\boldsymbol{a}_{s_1} := [1, 0, 0]^\top, \boldsymbol{a}_{s_2} := [0, 1, 0]^\top, \boldsymbol{a}_{s_3} := [0, 0, 1]^\top$$

- the matrices of coefficients of the variables in the constraints, for basic and non-basic variables: $\boldsymbol{B} \in \mathbb{R}^{m \times m}$, $\boldsymbol{N} \in \mathbb{R}^{m \times (n-m)}$

$\in \mathbb{R}^m$

$$\boldsymbol{B} := [\boldsymbol{a}_{s_1}, \boldsymbol{a}_{x_3}, \boldsymbol{a}_{x_1}] = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \qquad \boldsymbol{N} := [\boldsymbol{a}_{x_2}, \boldsymbol{a}_{s_2}, \boldsymbol{a}_{s_3}] = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}$$

s_1, x_3, x_1

Matrices in linear programming

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
	x_n			x_b			
x_b: s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	2
c	5	10	10	0	0	0	280

$$\begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = (0 \times 24) + (20 \times 8) + (60 \times 2)$$

$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top$, $\boldsymbol{x}_n := [x_2, s_2, s_3]^\top$, $\boldsymbol{c}_b := [0, 20, 60]^\top$, $\boldsymbol{c}_n := [30, 0, 0]^\top$

$\boldsymbol{a}_{x_1} := [8, 4, 2]^\top$, $\boldsymbol{a}_{x_2} := [6, 2, 1.5]^\top$, $\boldsymbol{a}_{x_3} := [1, 1.5, 0.5]^\top$,

$\boldsymbol{a}_{s_1} := [1, 0, 0]^\top$, $\boldsymbol{a}_{s_2} := [0, 1, 0]^\top$, $\boldsymbol{a}_{s_3} := [0, 0, 1]^\top$, $\boldsymbol{b} := [48, 20, 8]^\top$,

$$\boldsymbol{B} := [a_{s_1}, a_{x_3}, a_{x_1}] = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}, \boldsymbol{N} := [a_{x_2}, a_{s_2}, a_{s_3}] = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}, \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix}$$

Matrices in linear programming

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$c_b^T B^{-1} a_j - c_j$ $c_b^T B^{-1} N - c_n^T$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

	x_n: x_2	s_2	s_3	x_b: s_1	x_3	x_1	RHS
x_b: s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	2
c	5	10	10	0	0	0	280

$\boldsymbol{c}_b := [0, 20, 60]^\top$, $\boldsymbol{c}_n := [30, 0, 0]^\top$, $\boldsymbol{b} := [48, 20, 8]^\top$, $\boldsymbol{B}^{-1}\boldsymbol{b} = [24, 8, 2]^\top$.

$$\boldsymbol{B}^{-1}\boldsymbol{a}_j \implies \boldsymbol{B}^{-1}\boldsymbol{N} = \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix}, \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I},$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{a}_j - c_j \implies$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0] = [5, 10, 10],$$

$$\boldsymbol{c}_n^\top \boldsymbol{B}^{-1}\boldsymbol{B} - \boldsymbol{c}_n^\top = \boldsymbol{0}^\top,$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{b} = [0, 20, 60] \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = 280.$$

Cases for Sensitivity Analysis

Cases for sensitivity analysis

- **Sensitivity analysis** analyzes how much effect some change in something has on the optimization.
- We can have different cases of change in linear programming:
  1. change in **coefficient** of a variable (basic or nonbasic) in the **objective function**
     - ★ 1-1: change for **nonbasic** variable → c_j
     - ★ 1-2: change for **basic** variable
  2. change in **coefficient** of a variable (basic or nonbasic) in the **constraint(s)**
     - ★ 2-1: change for **nonbasic** variable
     - ★ 2-2: change for **basic** variable

     a_{ij}
  3. **adding** a new **variable** to optimization
  4. **adding** a new **constraint** to optimization

Note: we can have a **combination** of changes, too!

Case 1-1 of Change

Case 1-1 for sensitivity analysis

⋆ Change in **coefficient** of a **nonbasic** variable in the **objective function**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$
\begin{cases}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
 & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
 & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
 & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{cases}
$$

32

36

The company is able to increase the profit of the second product, $\boldsymbol{x}_2$, to (a) \$32 and (b) \$36. Do you recommend this change to the manager?

Case 1-1 for sensitivity analysis

⋆ Change in **coefficient** of a **nonbasic** variable in the **objective function**.

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$c_b^T B^{-1} a_j - c_j^T$ $c_b^T B^{-1} N - c_n^T$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

$$\mathbf{x}_b := [s_1, x_3, x_1]^\top, \mathbf{x}_n := [x_2, s_2, s_3]^\top, \mathbf{c}_b := [0, 20, 60]^\top, \mathbf{c}_n := [30, 0, 0]^\top.$$

$\mathbf{x}_2$ is a nonbasic variable. We have change in $c_{\mathbf{x}_2}$ in $\mathbf{c}_n$ so:

$$\mathbf{c}_b^\top B^{-1}\mathbf{a}_{\mathbf{x}_2} - c_{\mathbf{x}_2} = [0, 20, 60]\begin{bmatrix}-2\\-2\\5/4\end{bmatrix} - \overbrace{(30+\delta)}^{30+\delta} = 35 - 30 - \delta = 5 - \delta$$

column of x_2 in $B^{-1}N$
(first column of $B^{-1}N$)

For not having change in optimization:

$$5 - \delta \geq 0 \Longrightarrow \delta \leq 5 \Longrightarrow c_{\mathbf{x}_2,\text{new}} = 30 + \delta \leq 35.$$

30 ↓ 35

Case 1-1 for sensitivity analysis

- For not having change in optimization: $5-\delta \ge 0 \implies \delta \le 5 \implies c_{x_2,\text{new}} = 30+\delta \le 35$.
- Therefore, if profit of x_2 is $\$32 \le \$35$, we do not recommend it as it does not change the previous optimal solution for production of the company.
- If profit of x_2 is $\$36 > \$35$, we should continue the optimization:

		x_2	s_2	s_3	s_1	x_3	x_1	RHS	min test:
	s_1	-2	2	-8	1	0	0	24	—
	x_3	-2	2	-4	0	1	0	8	—
	x_1	$5/4$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	1	2	$\frac{2}{5/4}=\frac{8}{5}$
Previous c row $\rightarrow$	c	5	10	10	0	0	0	280	
updated $\rightarrow$	c	-1	10	10	0	0	0	280	

		x_2	s_2	s_3	s_1	x_3	x_1	RHS	
$r_1+\frac{8}{5}r_3$	s_1	0	1.2	-5.6	1	0	1.6	27.2	$\Rightarrow s_1^* = 27.2$
$r_2+\frac{8}{5}r_3$	x_3	0	1.2	-1.6	0	1	1.6	11.2	$\Rightarrow x_3^* = 11.2$
$\frac{4}{5}r_3$	x_2	1	-0.4	1.2	0	0	0.8	1.6	$\Rightarrow x_2^* = 1.6$
$r_4+\frac{4}{5}r_3$	c	0	9.6	11.2	0	0	0.8	281.6	$\Rightarrow c^* = \boxed{281.6}$

all nonnegative

compare!

Case 1-2 of Change

Case 1-2 for sensitivity analysis

⋆ Change in **coefficient** of a **basic** variable in the **objective function**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

The company is decreasing the profit of the first product, $\boldsymbol{x}_1$, to (a) \$58 and (b) \$30. Do you recommend this change to the manager?

Case 1-2 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1}a_j - c_j^\top}_{c_b^\top B^{-1} N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1} B - c_b^\top}$	$\boxed{c_b^\top B^{-1} b}$

X X

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_1$ is a basic variable. We have change in c_{x_1} in $\boldsymbol{c}_b$ so:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60+\delta] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0]$$

X

$$= [5 + 1.25\delta, 10 - 0.5\delta, 10 + 1.5\delta]$$

For not having change in optimization:

$$5 + 1.25\delta \geq 0 \implies \boxed{\delta \geq -4,} \quad 10 - 0.5\delta \geq 0 \implies \boxed{\delta \leq 20,} \quad 10 + 1.5\delta \geq 0 \implies \boxed{\delta \geq -6.6,}$$

$$\implies -4 \leq \delta \leq 20, c_{x_1} = 60 + \delta \implies 56 \leq c_{x_1} \leq 80.$$

Case 1-2 for sensitivity analysis

- For not having change in optimization: $56 \leq c_{x_1} \leq 80$.
- Therefore, if profit of $\boldsymbol{x}_1$ decreases to $\underbrace{\$58 \in [56, 80]}$, this decrease does not change the overall profit and it can be recommended.
- If profit of $\boldsymbol{x}_1$ is decreased to $\$30 < \$56$, we should continue the optimization:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 30] \begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 5/4 & -1/2 & 3/2 \end{bmatrix} - [30, 0, 0] = [-32.5, 25, -35],$$

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{b} = [0, 20, 30] \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = 220.$$

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{C_b^T B^{-1} a_j - C_j^T}_{C_b^T B^{-1} N - C_n^T}$	$\underbrace{0^T}_{C_b^T B^{-1} B - C_b^T}$	$C_b^T B^{-1} b$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS	min test:
s_1	-2	2	-8	1	0	0	24	—
x_3	-2	2	-4	0	1	0	8	—
x_1	5/4	-1/2	3/2	0	0	1	2	$2/(3/2) = 4/3$
c	5	10	10	0	0	0	280	
updated → c	-32.5	25	-35	0	0	0	220	

Case 1-2 for sensitivity analysis

s_1, x_3, x_1

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	-2	2	-8	1	0	0	24
x_3	-2	2	-4	0	1	0	8
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	2
C	5	10	10	0	0	0	280
C	-32.5	25	-35	0	0	0	220

min test:
—
—
$\frac{2}{3/2} = \frac{4}{3}$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	$14/3$	$-2/3$	0	1	0	$16/3$	$104/3$
x_3	$4/3$	$2/3$	0	0	1	$8/3$	$40/3$
s_3	$5/6$	$-1/3$	1	0	0	$2/3$	$4/3$
C	$-10/3$	$40/3$	0	0	0	$70/3$	$800/3$

min test:
$\frac{104/3}{14/3} = \frac{104}{14}$
$\frac{40}{4} = 20$
$\frac{4/3}{5/6} = \frac{24}{15}$

	x_2	s_2	s_3	s_1	x_3	x_1	RHS
s_1	0	$6/5$	$-28/5$	1	0	$8/5$	$136/5$
x_3	0	$6/5$	$-8/5$	0	1	$8/5$	$56/5$
x_2	1	$-2/5$	$6/5$	0	0	$4/5$	$8/5$
C	0	12	4	0	0	26	272

$\Rightarrow s_1^* = 136/5$
$\Rightarrow x_3^* = 56/5$
$\Rightarrow x_2^* = 8/5$
$\Rightarrow \boxed{C^* = 272}$

all nonnegative.

compare to 280

s_1, x_3, x_2

So, changing profit of $\boldsymbol{x}_1$ to \$30 decreases the total profit to \$272 from \$280.

Case 2-1 of Change

Case 2-1 for sensitivity analysis

a_{ij}

⋆ change in **coefficient** of a **nonbasic** variable in the **constraint(s)**.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

50

$C_2 \to C_N$

a_{ij} of x_2 $\to$ N

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

before

The company is changing the resources for $\boldsymbol{x_2}$ as $8x_1 + 5x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 4x_2 + 0.5x_3 \leq \$8$. Also, the company is changing the profit of that product to 50. What is your recommendation to the manager?

combination of changes

after

Case 2-1 for sensitivity analysis

	x_n^T	x_B^T	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^T B^{-1} a_j - c_j^T}_{c_b^T B^{-1} N - c_n^T}$	$\underbrace{0^T}_{c_b^T B^{-1} B - c_b^T}$	$c_b^T B^{-1} b$

c row $\longrightarrow$

c_n, a_{ij} for N

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [\overset{50}{\cancel{30}}, 0, 0]^\top.$$

$\boldsymbol{x}_2$ is a nonbasic variable. We have change in $a_{\boldsymbol{x}_2}$ in $\boldsymbol{N}$, and a change in $c_{\boldsymbol{x}_2}$ so:

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{a}_{\boldsymbol{x}_2} - c_{\boldsymbol{x}_2} = \underbrace{[0, 20, 60]}_{c_b^T} \underbrace{\begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}}_{a_{x_2}} - 50 = 10 \geq 0.$$

It does not change the optimal solution so it does not change the total profit.
If that would become negative, we should have continued the table!

Case 2-2 of Change

Case 2-2 for sensitivity analysis

⋆ change in **coefficient** of a **basic** variable in the **constraint(s)**. $\rightarrow a_{ij}$ for basic (in B)

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$
\begin{aligned}
\underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.
\end{aligned}
$$

The company is changing the resources for $\boldsymbol{x}_1$ as $5x_1 + 6x_2 + x_3 \leq \$48$, $5x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $x_1 + 1.5x_2 + 0.5x_3 \leq \$8$. What is your recommendation to the manager?

Case 2-2 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1} a_j - c_j^\top}_{c_b^\top B^{-1} N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1} B - c_b^\top \,\to\, 0}$	$c_b^\top B^{-1} b$

B

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

Previous $\boldsymbol{B}$ was: $\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$.

$\boldsymbol{x}_1$ is a basic variable. We have change in $a_{\boldsymbol{x}_1}$ in $\boldsymbol{B}$, so:

$$\star \quad \boldsymbol{c}_b^\top \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c}_n^\top = [0, 20, 60] \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1.5 & 5 \\ 0 & 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}.$$

We compute it. If any of the values becomes negative, we should continue the table; otherwise, the total profit does not change.

Case 3 of Change

Case 3 for sensitivity analysis

$c = 60x_1 + 30x_2 + 20x_3$

s.t. $\begin{cases} 8x_1 + 6x_2 + x_3 + s_1 = 48 \\ 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\ 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \\ 3x_1 + 5x_2 + 0x_3 + s_4 = 2 \end{cases}$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

⋆ **adding** a new **variable** to optimization.

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$x_4^* = 0$

$$\begin{aligned} \underset{x_1,x_2,x_3,s_1,s_2,s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & \begin{cases} 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \end{cases} \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

$x_4^* > 0$

$\{s_1, x_3, x_1\}$

The company is adding a new product $\boldsymbol{x}_4$ with profit (a) \$15 or (b) \$25, and the constraint coefficients $\boldsymbol{a} = [1, 1, 1]^\top$. What is your recommendation to the manager?

Case 3 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1}a_j - c_j^\top}_{c_b^\top B^{-1}N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1}B - c_b^\top}$	$c_b^\top B^{-1}b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_4$ is a nonbasic variable. We calculate its value in the last row of the table (if $c_{x_4} = 15$):

column in N

$$\underbrace{\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}}\boldsymbol{a}_{\boldsymbol{x}_4} - c_{\boldsymbol{x}_4} = \underbrace{\overbrace{[0, 20, 60]}^{c_b^\top} \overbrace{\begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}}^{B^{-1}}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 = 5 \geq 0.$$

It does not change the optimal solution so it does not change the total profit.

Case 3 for sensitivity analysis

	$x_n^\top$	$x_B^\top$	RHS
x_B	$B^{-1}a_j = B^{-1}N$	$B^{-1}a_j = B^{-1}B = I$	$B^{-1}b$
c	$\underbrace{c_b^\top B^{-1}a_j - c_j^\top}_{c_b^\top B^{-1}N - c_n^\top}$	$\underbrace{0^\top}_{c_b^\top B^{-1}B - c_b^\top}$	$c_b^\top B^{-1}b$

$$\boldsymbol{x}_b := [s_1, x_3, x_1]^\top, \boldsymbol{x}_n := [x_2, s_2, s_3]^\top, \boldsymbol{c}_b := [0, 20, 60]^\top, \boldsymbol{c}_n := [30, 0, 0]^\top.$$

$\boldsymbol{x}_4$ is a nonbasic variable. We calculate its value in the last row of the table (if $c_{\boldsymbol{x}_4} = 25$):

$$\boldsymbol{c}_b^\top \boldsymbol{B}^{-1}\boldsymbol{a}_{\boldsymbol{x}_4} - c_{\boldsymbol{x}_4} = [0, 20, 60]\begin{bmatrix}1 & 2 & -8\\ 0 & 2 & -4\\ 0 & -0.5 & 1.5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} - 25 = -5 < 0.$$

We should continue the table.

Case 3 for sensitivity analysis

$$B^{-1}a_{x_4} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

	x_2	s_2	s_3	s_1	x_3	x_1	x_4	RHS	min test:
s_1	-2	2	-8	1	0	0	-5	24	—
x_3	-2	2	-4	0	1	0	-2	8	—
x_1	5/4	-1/2	3/2	0	0	1	1	2	$2/1 = 2$
c	5	10	10	0	0	0	-5	280	

	x_2	s_2	s_3	s_1	x_3	x_1	x_4	RHS	
s_1	17/4	-1/2	-1/2	1	0	5	0	34	$\Rightarrow s_1^* = 34$
x_3	1/2	1	-1	0	1	2	0	12	$\Rightarrow x_3^* = 12$
x_4	5/4	-1/2	3/2	0	0	1	1	2	$\Rightarrow x_4^* = 2$
c	45/4	15/2	35/2	0	0	5	0	290	$\Rightarrow c^* = 290$

all nonnegative

So, the optimum objective function has increased and this addition of variable in beneficial.

Case 4 of Change

Case 4 for sensitivity analysis

$\max_{x_1, x_2} \; 3x_1 + 4x_2$

$\downarrow$

$x_1^* = x_2^* = \infty$

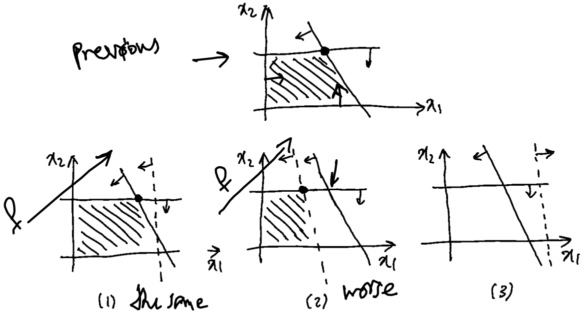
$\downarrow$

$c^* = \infty$

⋆ **adding** a new **constraint** to optimization.

This can result in three sub-cases:

- 4-1: The current optimal solution **satisfies** the new constraint.
- 4-2: The current optimal solution **doesn't satisfy** the new constraint but linear programming still **has a feasible solution**.
- 4-3: The current optimal solution **doesn't satisfy** the new constraint and linear programming **doesn't have a feasible solution**.



$f(x^*)$

Question: Can adding a constraint improve the optimum value of objective function?

c^*

Case 4-1 for sensitivity analysis

Example: The company has profits \$60, \$30, and \$20 for the first, second, and third products. The resources for these products have the following restrictions: $8x_1 + 6x_2 + x_3 \leq \$48$, $4x_1 + 2x_2 + 1.5x_3 \leq \$20$, and $2x_1 + 1.5x_2 + 0.5x_3 \leq \$8$.

$$\begin{aligned} \underset{x_1, x_2, x_3, s_1, s_2, s_3}{\text{maximize}} \quad & c = 60x_1 + 30x_2 + 20x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48, \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20, \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8, \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0. \end{aligned}$$

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint:

$$\boxed{x_1 + x_2 + x_3 \leq 11.}$$

It satisfies the current solution:

$$2 + 0 + 8 = 10 \leq 11 \quad \checkmark$$

Case 4-2 for sensitivity analysis

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint: $x_2 \geq 1$. It doesn't satisfy the current solution: $0 \ngeq 1$.
The new constraint: $x_2 + x_3 \leq 4$ $\quad x_2 + x_3 \geq 1$

$$\underbrace{x_2 \geq 1} \xrightarrow{\times -1} \underbrace{-x_2 \leq -1} \implies \underbrace{-x_2 + s_4 = -1}.$$

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	-2	2	-8	1	0	0	0	24
x_3	-2	2	-4	0	1	0	0	8
x_1	5/4	-1/2	3/2	0	0	1	0	2
s_4	-1	0	0	0	0	0	0	-1
C	5	10	10	0	0	0	0	280

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	0	2	-8	1	0	0	-2	26
x_3	0	2	-4	0	1	0	-2	10
x_1	0	-0.5	1.5	0	0	1	1.25	0.75
x_2	1	0	0	0	0	0	-1	1
C	10	10	0	0	0	0	5	275

all nonnegative

$s_1^* = 26, x_3^* = 10, x_1^* = 0.75, x_2^* = 1, C^* = 275$

Note: we have used the dual simplex method above.

Case 4-3 for sensitivity analysis

We saw in the table (see slide 8) that the solution is: $x_1^* = 2, x_2^* = 0, x_3^* = 8$.
The company is adding a new resource constraint: $x_1 + x_2 \geq 12$. It doesn't satisfy the current solution: $2 \not\geq 12$.
The new constraint:

$$x_1 + x_2 \geq 12 \xrightarrow{\times -1} -x_1 - x_2 \leq -12 \implies -x_1 - x_2 + s_4 = -12.$$

	x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
s_1	-2	2	-8	1	0	0	0	24
x_3	-2	2	-4	0	1	0	0	8
x_1	$5/4$	$-1/2$	$3/2$	0	0	1	0	2
s_4	-1	0	0	0	0	-1	1	-12
c	5	10	10	0	0	0	0	280

min test:
$\left|\frac{5}{-1}\right| = 5$
$\left|\frac{0}{-1}\right| = 0$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
	s_1	-2	2	-8	1	0	0	0	24
	x_3	-2	2	-4	0	1	0	0	8
$r_3 + r_4$	x_1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	0	0	-10
$-r_4$	x_1	1	0	0	0	0	1	-1	12
	c	5	10	10	0	0	0	0	280

min test:
$\left|\frac{10}{-\frac{1}{2}}\right| = 20$

Note: we have used the dual simplex method above.

Case 4-3 for sensitivity analysis

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
	s_1	-2	2	-8	1	0	0	0	24
	x_3	-2	2	-4	0	1	0	0	8
r_3+r_4	x_1	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	0	0	-10
$-r_4$	x_1	1	0	0	0	0	1	-1	12
	c	5	10	10	0	0	0	0	280

min test: $\left|\frac{10}{-\frac{1}{2}}\right| = 20$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
r_1+4r_3	s_1	-1	0	-2	1	0	0	4	-16
r_2+4r_3	x_3	-1	0	2	0	1	0	4	-32
$-2r_3$	s_2	$-\frac{1}{2}$	1	-3	0	0	0	-2	20
r_4	x_1	1	0	0	0	0	1	-1	12
r_5+20r_3	c	10	0	40	0	0	0	20	80

min test: $\left|\frac{10}{-1}\right| = 10$

		x_2	s_2	s_3	s_1	x_3	x_1	s_4	RHS
r_1-r_2	s_1	0	0	-4	1	-1	0	0	16
$-r_2$	x_2	1	0	-2	0	-1	0	-4	32
$r_3-\frac{1}{2}r_2$	s_2	0	1	-4	0	-0.5	0	-4	36
r_4+r_2	x_1	0	0	2	0	1	1	3	-20
r_5+10r_2	c	0	0	60	0	10	0	60	-240

no negative entry in row!

Therefore, it does not have a feasible solution!

Acknowledgment

This lecture is inspired by the lectures of Prof. Shokoufeh Mirzaei on sensitivity analysis in linear programming: [Link]