Fully Connected Neural Network

Deep Learning (ENGG*6600*01)

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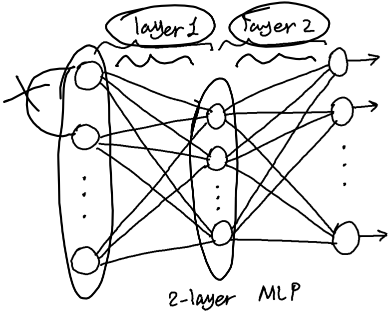
MLP

- A **fully connected neural network** is a stack of layers of neural network where in every layer, all the neurons of the previous layer are connected to all the neurons of the next layer.
- Every layer of the fully connected neural network is called a **fully connected layer** or a **dense layer**.
- Each neuron in the fully connected neural network is a Perceptron neuron. That is why this network is also called the **Multi-Layer Perceptron (MLP)**.
- MLP was proposed by **Rosenblatt** in 1958, in the same paper as Perceptron [1]. In that paper, he proposed an MLP with three layers.



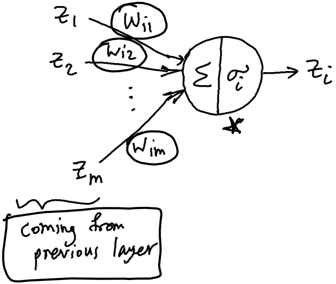
Neuron

- Each neuron in the fully connected neural network is a Perceptron neuron. So it has:
  - Summation of the outputs of previous layer multiplied by the weights of previous layer:

$$a_i = \sum_{\ell=1}^{m} w_{i\ell} z_\ell. \tag{1}$$

  - Activation function:

$$z_i := \sigma_i(a_i). \tag{2}$$

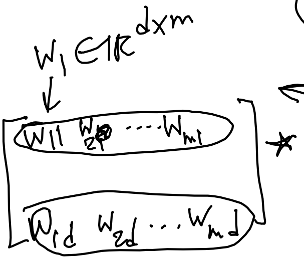


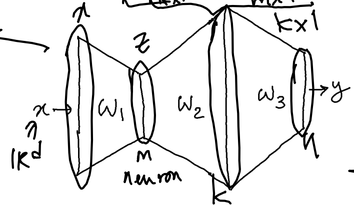
Layer as a Projection

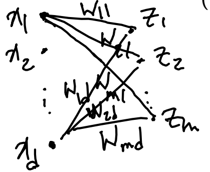
$W_1 \in \mathbb{R}^{d \times m}, \; W_2 \in \mathbb{R}^{m \times k}, \; W_3 \in \mathbb{R}^{k \times \eta}$

- Every layer in the fully connected network can be seen as a linear projection followed by an activation function.

$$\boldsymbol{y} = \sigma_3\Big(\overset{\eta \times k}{\boldsymbol{W}_3^\top}\,\sigma_2\big(\underbrace{\boldsymbol{W}_2^\top}_{k \times m}\,\underbrace{\sigma_1(\overset{m \times d}{\boldsymbol{W}_1^\top}\,\overset{\mathbb{R}^d}{\boldsymbol{x}})}_{m \times 1}\big)\Big). \tag{3}$$



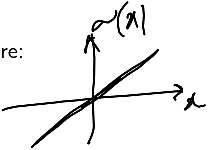




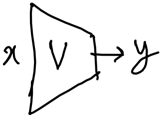
- The activation function is usually a nonlinear function because if all activation functions are linear in the network, the entire network is collapsed to be one linear projection.

$$\boldsymbol{y} = \underbrace{\boldsymbol{W}_3^\top \boldsymbol{W}_2^\top \boldsymbol{W}_1^\top}_{V^\top} \boldsymbol{x} = \boldsymbol{V}^\top \boldsymbol{x}, \tag{4}$$

where:



$$\boxed{\boldsymbol{V} := \boldsymbol{W}_1 \boldsymbol{W}_2 \boldsymbol{W}_3.} \tag{5}$$



Activation Function

There exist various activation functions. Some of them are:

- Linear (identity) function:

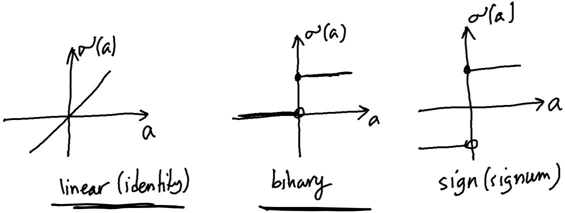
$$\sigma(a) = a, \quad \sigma(a) \in (-\infty, \infty), \quad \sigma'(a) = 1. \tag{6}$$

- Binary step:

$$\sigma(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1 & \text{if } a \geq 0 \end{cases}, \quad \sigma(a) \in \overset{\{0,1\}}{\cancel{[0,1]}}, \quad \sigma'(a) = \begin{cases} 0 & \text{if } a \neq 0 \\ \text{undefined} & \text{if } a = 0. \end{cases} \tag{7}$$

- Sign (signum) function:

$$\sigma(a) = \begin{cases} -1 & \text{if } a < 0 \\ 1 & \text{if } a \geq 0 \end{cases}, \quad \sigma(a) \in \overset{\{-1,1\}}{\cancel{[-1,1]}}, \quad \sigma'(a) = \begin{cases} 0 & \text{if } a \neq 0 \\ \text{undefined} & \text{if } a = 0. \end{cases} \tag{8}$$



Activation Function

- Logistic (sigmoid) function:

$$\boxed{\sigma(a) = \frac{1}{1+e^{-a}},} \quad \sigma(a) \in [0,1], \quad \sigma'(a) = \sigma(a)(1-\sigma(a)). \tag{9}$$

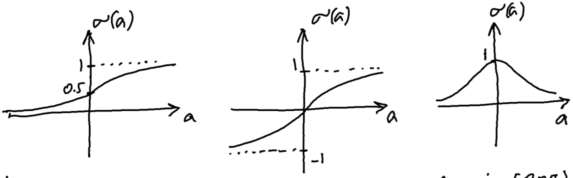
- Hyperbolic tangent (tanh):

$$\boxed{\sigma(a) = \frac{e^{a} - e^{-a}}{e^{a}+e^{-a}},} \quad \sigma(a) \in [-1,1], \quad \sigma'(a) = 1-\sigma(a)^2. \tag{10}$$

- Gaussian (radial basis function):

$e^{-(a-\mu)^2}$

$$\boxed{\sigma(a) = e^{-a^2},} \quad \sigma(a) \in (0,1], \quad \sigma'(a) = -2ae^{-a^2}. \tag{11}$$



logistic (sigmoid) tanh Gaussian (RBF)

Activation Function

- Softplus [2] (2011):

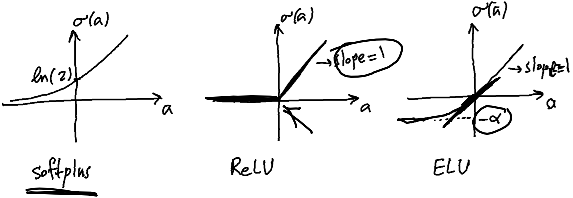
$$\boxed{\sigma(a) = \ln(1 + e^{a}),} \quad \sigma(a) \in [0, \infty), \quad \sigma'(a) = \frac{1}{1 + e^{-a}}. \tag{12}$$

- Rectified linear unit (ReLU) [3] (2010):

$$\boxed{\sigma(a) = \begin{cases} 0 & \text{if } a < 0 \\ a & \text{if } a \geq 0 \end{cases}} = \boxed{\max(0, a),} \quad \sigma(a) \in [0, \infty), \quad \sigma'(a) = \begin{cases} 0 & \text{if } a < 0 \\ \text{undefined} & \text{if } a = 0 \\ 1 & \text{if } a > 0. \end{cases} \tag{13}$$

- Exponential Linear Unit (ELU) [4] (2015):

$$\sigma(a) = \begin{cases} \alpha(e^{a} - 1) & \text{if } a < 0 \\ a & \text{if } a \geq 0 \end{cases}, \quad \sigma(a) \in (-\alpha, \infty), \quad \sigma'(a) = \begin{cases} \alpha e^{a} & \text{if } a < 0 \\ 1 & \text{if } a = 0, \alpha = 1 \\ 1 & \text{if } a > 0. \end{cases} \tag{14}$$



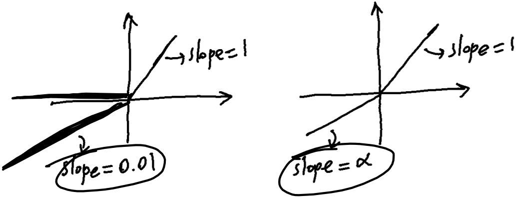
Activation Function

- Leaky rectified linear unit (Leaky ReLU) [5] (2013):

$$\sigma(a) = \begin{cases} 0.01a & \text{if } a < 0 \\ a & \text{if } a \geq 0 \end{cases}, \quad \sigma(a) \in (-\infty, \infty), \quad \sigma'(a) = \begin{cases} 0.01 & \text{if } a < 0 \\ \text{undefined} & \text{if } a = 0 \\ 1 & \text{if } a > 0. \end{cases} \tag{15}$$

- Parametric rectified linear unit (PReLU) [6] (2015):

$$\boxed{\sigma(a) = \begin{cases} \alpha a & \text{if } a < 0 \\ a & \text{if } a \geq 0 \end{cases}}, \quad \sigma(a) \in (-\infty, \infty), \quad \sigma'(a) = \begin{cases} \alpha & \text{if } a < 0 \\ \text{undefined} & \text{if } a = 0 \\ 1 & \text{if } a > 0. \end{cases} \tag{16}$$



Activation Function

$a_1, \dots, a_m \rightarrow \frac{a_1}{a_1+\dots+a_m}, \dots, \frac{a_m}{a_1+\dots+a_m}$

- Softmax:

$$\boxed{\sigma_i(\mathbf{a}) = \frac{e^{a_i}}{\sum_{i=1}^{m} e^{a_i}}}, \forall i \in \{1, \dots, m\}, \quad \sigma_i(\mathbf{a}) \in (0,1), \quad \sigma'(a) = \sigma_i(\mathbf{a})\big(\delta_{ij} - \sigma_j(\mathbf{a})\big), \quad (17)$$

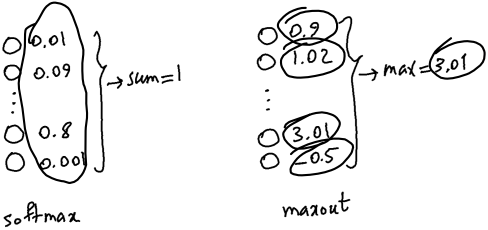
where δ_{ij} is the Kronecker delta:

$[\sigma_1(\mathbf{a}), \dots, \sigma_m(\mathbf{a})]$

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \quad (18)$$

- Maxout [7] (2013):

$$\sigma_i(\mathbf{a}) = \max_j(\sigma_j), \quad \sigma_i(\mathbf{a}) \in (-\infty, \infty), \quad \sigma'(a) = \begin{cases} 1 & \text{if } i = \arg\max_j(\sigma_j) \\ 0 & \text{if } i \neq \arg\max_j(\sigma_j). \end{cases} \quad (19)$$



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